Differential Geometry II

D-MATH, FS 2015

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Solution of Exercise Sheet 4

1. The twisting number is -1.

2. (a) Recall that $L_X Y = [X, Y]$. But the bracket isn't linear over $C^\infty(M)$, i.e

$$[fX, Y] = f[X, Y] - (Y \cdot f) X.$$

Then $L_{(-)}^{(-)} = [-, -] : TM \times TM \to TM$ is not an affine connection.

(b) Set $Z := g(x, y)\frac{\partial}{\partial x}$, $W := f(x, y)\frac{\partial}{\partial x}$ with $\left(\frac{\partial}{\partial x}g\right)(x, 0) = 0$ and $f(x, 0) = 1$. Then

$$
\begin{aligned}
L_Z^W(x, 0) &= \left[g(x, 0)\frac{\partial}{\partial x}, f(x, 0)\frac{\partial}{\partial y}\right] \\
&= g(x, 0) f(x, 0)\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] + g(x, 0)\left(\left(\frac{\partial}{\partial x}f\right)(x, 0)\right)\frac{\partial}{\partial y} - f(x, 0)\left(\left(\frac{\partial}{\partial y}g\right)(x, 0)\right)\frac{\partial}{\partial x}.
\end{aligned}
$$

Note that the first term vanish. So for example we can choose $f(x, y) = 1$ and $g(x, y) = y + 1$.

3. The fact that D_X^F is a connection on the subbundle F is straightforward: since π_F is a bundle map then it is fiber wise linear, i.e for any smooth section V defined on $U \subset M$ and a smooth map $f : U \to \mathbb{R}$ we have

$$\pi_F(f \cdot V) = f \cdot \pi_F(V)$$

D_X^F satisfies the two linear condition, the Leibniz rule is obtained in a similar way. Since π_F is fiber wise orthogonal, we can decompose each fiber as the orthogonal sum

$$E_p = F_p \oplus \ker \pi_F(p)$$

where $\pi_F(p) : E_p \to F_p$ is the orthogonal projection. Since all the data are smooth it is easy shown by linear algebra that $\ker \pi_F$ is another vector subbundle of E and there is a bundle map

$$\pi_2 : E \to \ker \pi_F$$

point wise defined as the projection in the second element in the above direct sum. Next observe that we can consider a section of F (resp. $\ker \pi_F$) as a section of E, and that each section V on E decomposes to a section $\pi_F(V)$ on E and a section $\pi_2(V)$ on $\ker \pi_F$. With these identifications in place, we can write the equation on sections V of E, $V = \pi_2(V) + \pi_F(V)$. Note that a connection doesn't preserves the

orthogonal decomposition in general, i.e let V be a section of $F \subset E$ and W a section of $\ker \pi_F \subset E$. Then

$$0 = X \cdot < V, W >_h = < D_X V, W >_h + < V, D_X W >_h$$

but it is not true in general that $< D_X V, W >= 0$, i.e. $D_X V \in F \subset E$. By the above discussion we know that for a section V on $F \subset E$ we get two sections $\pi_F(D_X V), \pi_2(D_X V)$. Then for another section W of F we have

$$< D_X V, W >=< \pi_F(D_X V) + \pi_2(D_X V), W >_h = < \pi_F(D_X V), W >_h .$$

It follows that D_X^F is compatible.

4. (a) By definition of V parallel section of E, we have for every vector field $X \in \Gamma(TM)$, that

$$D_X V = 0.$$

Thus from metric compatibility, we have:

$$X \cdot \langle V, V \rangle = 2 \langle D_X V, V \rangle = 0.$$

Now this then of course means $x \rightarrow \langle V(x), V(x) \rangle$ is constant. (If you don't see this, go to local coordinates and see that this implies all the partial differentials to be zero, and thus the conclusion, locally, but then implicitly, we assume all the time, that the base manifold M is connected, otherwise, you have this function constant on every connected component.)

(b) Let $U \subset M$ sufficiently small such that there exist an orthonormal frame e_α. Then for any two section V, W there exists smooth function f_α, g_α on U such that

$$V = f_\alpha e_\alpha, \quad W = g_\alpha e_\alpha$$

Since the connection is compatible we have have for any $X \in TM$

$$X < V, W >=< D_X V, W > + < V, D_X W > .$$

We show

$$X < V, W >=< D_X^0 V, W > + < V, D_X^0 W >$$

Note that

$$< V, W >= f_\alpha g_\alpha, \; < D_X^0 V, W >= X(f_\alpha) g_\alpha , < V, D_X^0 W >= f_\alpha X(g_\alpha)$$

So the above equation follow by the Leibniz rule for vectorfields.