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Solution of Exercise sheet 5

1. (a) Let $U \subset M$ be open and let $X, Y \in C^\infty(U, TM)$, $V, W \in C^\infty(U, E)$ then by a direct computation we have

$$\Delta(X+Y, V) = \Delta(X, V) + \Delta(Y, V), \quad \Delta(X, W+V) = \Delta(X, W) + \Delta(X, V)$$

Now let $f \in C^\infty(U)$, then

$$\Delta(fX, V) = D_{fX^2}V - D_{fX^1}V = fD^2_X V - fD^1_X V = f\Delta(X, V).$$

On the other hand

$$\begin{aligned}\Delta(X, fV) &= D^2_X fV - D^1_X fV \\ &= fD^2_X V + (Xf)V - \big(fD^1_X V + (Xf)V\big) \\ &= f\Delta(X, V)\end{aligned}$$

This show that $\Delta(X, V)$ is a bilinear map that is linear over $C^\infty(M)$.

Now it is a very important concept of differential geometry that $C^\infty(M)$ linear maps are 'tensors', i.e. they come from sections of certain tensor bundles. (in our case $TM^* \otimes E^* \otimes E = \mathrm{Bilin}(TM, E; E)$.) To see this we will use two steps.

(i) Δ operates locally. That is, given $p \in M$, U an open neighborhood of p, $\chi : M \to [0, 1]$ a cut-off function such that $\mathrm{supp}(\chi) \subset U$ and $\chi(p) = 1$ and X, Y 2 vector fields that agree on U, then we have

$$\begin{aligned}\Delta(X, V)(p) - \Delta(Y, V)(p) &= \chi(p)(\Delta(X, V)(p) - \Delta(Y, V)(p)) \\ &= (\chi(\Delta(X - Y, V)))(p) = (\Delta(\chi(X - Y), V))(p) \\ &= \Delta(0, V)(p) = 0\end{aligned}$$

where we used $\chi(X - Y) \equiv 0$. The same trick can be applied to the second component.

(ii) Δ operates point wise. Take $U \subset M$ a common trivializing neighborhood around $p \in M$ for the vector bundles E and TM. Now we can take frames $e_1, \ldots, e_n$ reps. $f_1, \ldots, f_m$ for $TM|_U$ resp. $E|_U$. Let $X|_U = \sum_i X^i e_i$ and $V|_U = \sum_j V^j f_j$ be sections of the respective vector bundles over M. Up to the abuse of notation, were we omit multiplying everything with χ, we get

$$\begin{aligned}\Delta(X,V)(p) &= \Delta(\sum_i X^i e_i, \sum_j V^j f_j)(p)\\ &= (\sum_{i,j} X^i V^j \Delta(e_i, f_j))(p)\\ &= \sum_{i,j} X^i(p) V^j(p) \Delta(e_i, f_j)(p) = \sum_{i,j,k} X^i(p) V^j(p) D^k_{i,j}(p) f_k\end{aligned}$$

where $\Delta(e_i, f_j)(p) := D^k_{i,j}(p) f_k$ Thus we defined

$$\Delta(p) : T_pM \times E_p \to E_p$$

which can be verified to be independent of the frame chosen. These $\Delta(p)$ also vary smoothly as the $D^k_{i,j}$ are smooth. Thus these form a section of $\operatorname{Bilin}(TM, E; E)$.

(b) Let $\Delta(-,-)$ be a section of $\operatorname{Bilin}(TM, E; E)$, i.e it is a smooth family of bilinear maps $\Delta_p(-,-) : T_pM \times E_p \to E_p$. Then for any smooth functions f, g any two scalars

$$\begin{aligned} D^2_{fX+gY}V &= D^1_{fX+gY}V + \Delta(fX+gY, V)\\ &= D^1_{fX}V + D^1_{gY}V + \Delta(fX, V) + \Delta(gY, V)\\ &= D^2_{fX}V + D^2_{gY}V,\end{aligned}$$

and analogously $D^2_X(aV + bW) = aD^2_X(V) + bD^2_X(W)$. The last remaining step is the Leibniz rule.

$$\begin{aligned} D^2_X(fV) = D^1_X(fV) + \Delta(X, fV) &= fD^1_XV + X(f)V + f\Delta(X,V)\\ &= fD^2_XV + X(f)V.\end{aligned}$$

(c) This is a direct consequence of the two statements.

2. Let E be a vectorbundle on M. Consider the section V defined by

$$V(p) = 0 \text{ for each } p \in M$$

it is called the *zero section.* Is it easy show that the zero section is actually smooth section. Moreover, by linearity, is it necessarily parallel: for any connection D on E

$$D_XV = D_X 0V = 0 D_XV = 0$$

Note that this is true for any vector bundle, so this section is also called the *trivial parallel section.* It isn't true that any bundle carries a non trivial parallel section. However this is true in the case of a trivial line bundle. Let E be a trivial line bundle, then it has an (ortho)normal frame given by the smooth map

$$e : M \to E$$

such that $e(p)$ is a basis for E_p (note that this is equivalent to say that $e(p)$ is never equal to 0). Now since D is compatible with $\langle -,-\rangle$ for any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} X\langle \lambda e, \lambda e\rangle &= 2\langle D_X \lambda e, \lambda e\rangle \\ &= 2\lambda^2 \langle D_X e, e\rangle \end{aligned}$$

We have two cases. Assume that $\lambda = 0$, then λe is the zero section and thus is parallel. If $\lambda \neq 0$ we get $\langle \lambda e, \lambda e\rangle$ constant, so

$$\langle D_X e, e\rangle = 0$$

which implies $D_X e = 0$.

(a) Use the formula for the Christoffel symbols of the Levi-Civita connection seen in class and compute.

(b) $Y(t) = (f^1(t), f^2(t))$ be the solution of

$$\frac{D}{dt}Y = 0, \quad Y(0) = Y_0 \tag{1}$$

along γ. Note that Y is a section defined on the image of γ. Then the first component of equation (1) is given by

$$\frac{d}{dt}f^2(t) + \sum_{i,j} \dot{\gamma}^i(t) f^j(t) \Gamma^1_{i,j} = 0$$

which reduces to

$$\frac{d}{dt}f^1(t) - f^2(t) = 0.$$

Analogously for the second component, we have

$$\frac{d}{dt}f^1(t) + \sum_{i,j} \dot{\gamma}^i(t) f^j(t) \Gamma^2_{i,j} = 0$$

which gives

$$\frac{d}{dt}f^2(t) + f^1(t) = 0.$$

Since $(f^1(0), f^2(0)) = (1,0)$ we conclude that

$$f^1(t) = \cos(t), \quad f^2(t) = \sin(t)$$

(c) Follows from the above.

3. (a) Since E is a complex line bundle (over $M = \mathbb{R}^2$) the fiber wise multiplication by i induces a bundle map $m_i : E \to E$ sending (p,z) to (p, iz).
On the other hand E may be identified as a $\mathbb{R}^2$ bundle, we denote this bundle with $E_{\mathbb{R}}$. Analogously the multiplication with i may be identified with a bundle map $m_i : E_{\mathbb{R}} \to E_{\mathbb{R}}$ such that $m_i^2 = -id_E$ and $\langle -,-\rangle$ induces a m_i invariant real inner product $\langle -,-\rangle_{\mathbb{R}}$ on each fiber of $E_{\mathbb{R}}$. Now since $M = \mathbb{R}^2$ is contractible we get that $E_{\mathbb{R}}$ is a trivial vector bundle, and in particular it carries a unit length section f_1. Now set $e_1 := f_1$, $f_2 := m_i(e_1)$. We have

$$\langle f_1, f_2\rangle_{\mathbb{R}} = \langle e_1, m_i(e_1)\rangle_{\mathbb{R}} = \langle m_i e_1, -(e_1)\rangle_{\mathbb{R}} = -\langle f_1, f_2\rangle_{\mathbb{R}}.$$

Thus $m_i(e_1)$ is orthogonal to e_1. By abuse of notation, the remaining part of the exercise we will denote m_i with i, $E_{\mathbb{R}}$ with E and $\langle -,-\rangle_{\mathbb{R}}$ with $\langle -,-\rangle$.

(b) By the above exercise we know that any connection D can be written as

$$D = D^0 + \Delta$$

where D^0 is take with respectt to e_1, e_2. Take $X \in \mathbb{C}^\infty(TM)$. Since D is compatible with the inner product, by exercise 4 of exercise sheet 4, we have that the map

$$\Delta_X(p) \; : \; E_p \to E_p$$

is anti-symmetric with respect to $\langle \cdot, \cdot \langle$. More precisely from

$$\langle \Delta_X e_1, e_1 \rangle + \langle e_1, \Delta_X e_1 \rangle = 0$$

we have $\langle \Delta_X e_1, e_1 \rangle = 0$. Thus there exists a $\lambda_X \in C^\infty(M)$ such that

$$\Delta_X e_1 = -\lambda_X e_2 = -i\lambda_X e_1.$$

On the other hand for e_2 we have that there exists a $\mu_X \in C^\infty(M)$ such that

$$\Delta_X e_2 = -i\mu_X e_2.$$

Since

$$\langle \Delta_X e_1, e_2 \rangle + \langle e_1, \Delta_X e_2 \rangle = 0$$

we have $\lambda_X = \mu_X$. Thus we can define

$$\omega : C^\infty(TM) \to C^\infty(M) : X \mapsto \lambda_X$$

If $f, g \in C^\infty(M)$, $V \in C^\infty(E)$ and $X, Y \in C^\infty(M)$ then

$$\begin{aligned} -i\lambda_{fX+gY} V &= \Delta(fX + gY, V) = f\Delta(X, V) + g\Delta(Y, V) \\ &= -i(f\lambda_X + g\lambda_Y)V. \end{aligned}$$

As V is arbitrary, one deduces

$$\lambda_{fX+gY} = f\lambda_X + g\lambda_Y$$

that is to say that ω is linear over $C^\infty(M)$ which as in the first exercise will lead to a section of a bundle, in this case TM^*.
(In plain English this would be ω eats vector and spits out a number.)
It follows that the multiplication $-i\omega(-) \; : \; T_pM \times E_p \to E_p$ is a smooth family of bilinear maps and we conclude that the connection has the wanted form

$$D^0 - i\omega(-).$$

(c) Since $D = D^0 - i\omega$ we have two facts. Clearly

$$D^0_X iV = iD^0_X V.$$

On the other hand note that

$$-i\omega(X)(ie_1) = -i\omega(X) \cdot (ie_1) = i\left(-i\omega(X) \cdot (e_1)\right)$$

and

$$-i\omega(X)(ie_2) = -i\omega(X) \cdot (ie_2) = i\left(-i\omega(X) \cdot (e_2)\right)$$

then $-i\omega(X)(iV) = i\left(-i\omega(X)(V)\right)$.

(d) Let $X = \partial_x, Y = \partial_y$ be the standard frame on TM. It induces a dual standard frame on TM^* which is denote by dx and dy, i.e. if $X = X^1\partial_x + X^2\partial_y$ then $dx(X) = X^1$ and $dy(X) = X^2$.
Let $V = e^{if}e_1$, $f \in C^\infty(M)$ be a section. Then

$$D^0_X V = D^0 e^{if} e_1 = i\partial_x(f) e^{if} e_1$$
$$D^0_Y V = D^0 e^{if} e_1 = i\partial_y(f) e^{if} e_1$$

and

$$-i\omega(\partial_x) V = -i(\omega(\partial_x)) e^{if} e_1$$
$$-i\omega(\partial_x) V = -i(\omega(\partial_y)) e^{if} e_1$$

Thus V is parallel if and only if

$$\partial_x(f) = (\omega(\partial_x))$$
$$\partial_y(f) = (\omega(\partial_y))$$

Let us now define the exterior derivative $d : C^\infty(M) \to C^\infty(TM^*)$ defined by

$$df = \partial_x f dx + \partial_y f dy.$$

Note that the above equalities can now be rewritten as

$$df = w$$

(e) Let $V = f_1e_1 + f_2e_2$. If V is parallel then

$$\partial_x f_1 e_1 + \partial_x f_2 e_2 = a(x,y) f_1 e_2 + a(x,y) f_2 e_1$$
$$\partial_y f_1 e_1 + \partial_y f_2 e_2 = b(x,y) f_1 e_2 + b(x,y) f_2 e_1$$

which is equivalent to

$$\partial_x f_1 = a(x,y) f_2$$
$$\partial_y f_2 = b(x,y) f_1$$
$$\partial_x f_2 = a(x,y) f_1$$
$$\partial_y f_1 = b(x,y) f_2$$

By identifying mixed derivatives, we get

$$\partial_y a(x,y) f_2 = \partial_x b(x,y) f_2$$
$$\partial_y a(x,y) f_1 = \partial_x b(x,y) f_1$$

and this gives the desired result as f_1,f_2 are never both zero at any point as V has constant non zero norm at every point.
For the converse, we set $V = e^{if}e_1$ and soon in the course (DeRahm cohomology in $\mathbb{R}^n$) you will see that the condition $df = \omega$ can be met by some f as $M = \mathbb{R}^2$ and $d\omega := (\partial_y a(x,y) - \partial_x b(x,y)) dx \wedge dy = 0$.

4. (a) We can look at (N, h) as being a submanifold of M such that the metric g restricts to h on $TN \subset TM$. It can be shown that the normal bundle $(TN)^\perp$ is also a vector sub bundle of $TM|_N$ as is TN and from this decomposition

$$TM|_N = TN \oplus (TN)^\perp$$

we also get an orthogonal projection $\pi : TM|_N \to TN$. (If you don't feel comfortable with this, try to trivialize the $(TN)^\perp$ by using for example Gram-Schmidt)

(b) Metric compatibility was already shown in Exercise 3 of Exercise Sheet 4.

(c) We have two facts. First,

$$[X, Y] = D_X Y - D_Y X$$

and second,

$$X, Y \in C^\infty(TN) \Rightarrow [X, Y] \in C^\infty(TN).$$

Therefore,

$$D^\pi_X Y - D^\pi_Y X = \pi\,(D_X Y - D_Y X) = \pi([X, Y]) = [X, Y]\,.$$

for $X, Y \in C^\infty(TN)$.

(d) As the Levi-Civita connection is the unique torsion free and metric connection on TN, D^π has to be the Levi-Civita connection.