# Examplesheet 6: Crashcourse on surfaces

1. (a) Show using an explicit map that the crosscap is homeomorphic to the Möbius strip.

   (b) Show that the Klein bottle is homeomorphic to the union of two copies of a Möbius strip joined (by a homeomorphism) along their boundaries.

2. (a) Show that a disc with a twisted handle attached is homeomorphic to a disc with two crosscaps attached.

   (b) Show that a disc with a crosscap and a handle attached is homeomorphic to a disc with three crosscaps attached.

3. (This exercise gives an alternative way of looking at the addition of a handle or twisted handle).

   Let $E$ be the disc of radius 10 in $\mathbb{C}$ minus the open unit discs centered at $z = \pm 5$. Let $X$ be the space $E \cup (S^1 \times I)$, where the cylinder is attached via $(e^{i\theta}, 0) \sim -5 + e^{i\theta}$ and $(e^{i\theta}, 1) \sim 5 + e^{-i\theta}$. Let $Y$ be the identification with $-5 + e^{i\theta} \sim 5 + e^{-i\theta}$.

   (a) Prove explicitly that $X$, which is the disc with a handle added is homemorphic to $Y$.

   (b) What happens when the $e^{-i\theta}$'s are replaced with $e^{i\theta}$'s

4. Let $F'$ be the space obtained by cutting a surface $F$ along a curve $C$. Prove the following lemma: There is a continuous "regluing" map $p : F' \to F$. The boundary of $F'$ is $\partial F' = p^{-1}(C) \sqcup \partial F$, and the new part $p^{-1}(C)$ consists of either one or two circles.

5. (a) Show that cutting along a separating curve increases the number of components of a surface by 1.

   (b) Show that cutting along a one-sided curve cannot separate a surface.

6. (a) For $X$ a combinatorial surface which can be written as a combinatorial union of sub-surfaces $A$ and $B$, $X = A \cup B$, show that

   $$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

   (b) The Euler characteristic of any combinatorial circle is 0.

7. Let $F$ be a combinatorial surface and let $C \subset F$ be a closed curve containing of vertices and edges.

   (a) Cut $F$ along the curve $C$ to obtain the surface $F'$ and show that

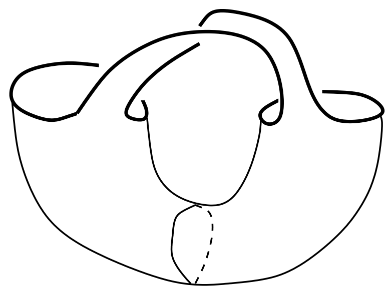
   $$\chi(F) = \chi(F')$$

(b) Instead of cutting along $C$ only, you can also cut and then cap off each boundary component (how many are there?) arising by gluing disc(s) onto it. This process is called "surgery on $C$". Find a formula for the Euler characteristic of the resulting surface.

8. Suppose that a connected surface $F$ is made by starting with $d$ closed discs and attaching $b$ bands to them.

   (a) Find a formula for the Euler characteristic of $F$ in terms of $b$ and $d$.

   (b) Identify the following surface.



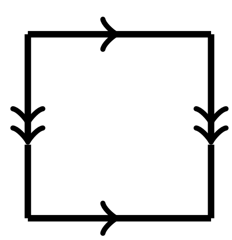
9. Find two triangulations of the 2-sphere. Overlap them and find a third triangulation that "contains" both of them. Check that they all yield the same Euler characteristic.

10. Let $F_1$ and $F_2$ be two surfaces. Show that

$$\chi(F_1 \# F_2) = \chi(F_1) + \chi(F_2) - 2$$

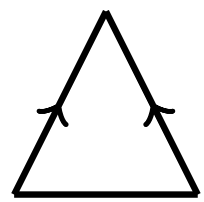
and deduce what this means for the genus of the connected sum of surfaces. Here the connected sum of two surfaces is defined by removing two discs from the surfaces and gluing them along their boundary (why is this operation well-defined?).

11. For $g \geq 0$ we denote by $\sum_g$ the closed orientable surface of genus $g$. As you know the torus $T = \sum_1$ can be obtained by glueing opposite sides of a square:



We say that this is a *planar model* for $T$. More generally, a planar model for a surface consists of a polygonal region together with glueing information that says how the sides of the polygon are identified to obtain the surface. It is not necessary that all sides are glued, which is to say that the resulting surface may have a non-empty boundary.

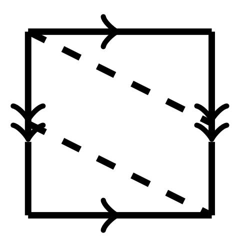
(a) Which surface does the following model represent?



(b) Find a planar model for the sphere $S^2 = \sum_0$.

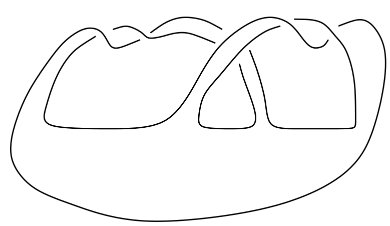
(c) Find a planar model for the surface $\sum_2$. Hint: Use the fact that $\sum_2 = \sum_1 \# \sum_1$.

(d) Convince yourself that the dashed lines in the following model form a simple closed curve on the torus. Draw the torus with the curve on it.



(e) By using another configuration of straight lines in the above model, describe the trefoil knot as a curve on $T \subset \mathbb{R}^3$.

12. Consider the following surface with boundary

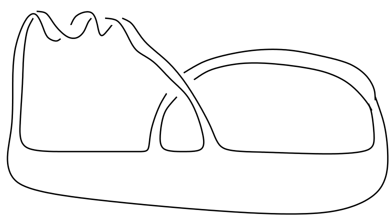


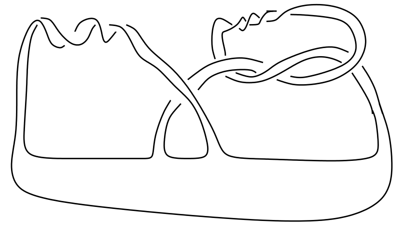
(a) Show that the boundary is a trefoil.

(b) Show that surface is homeomorphic to the same surface with the bands untwisted.

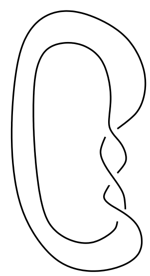
(c) Show that the above two surfaces are not isotopic (use the boundary).

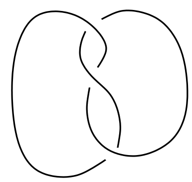
13. Describe in words a homeomorphism that takes the surface on the left to the surface on the right.



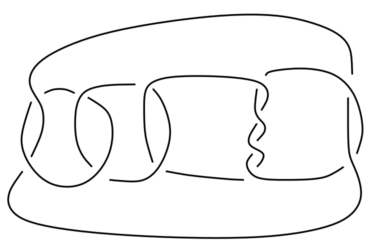


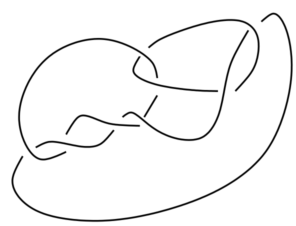
14. Identify the following knots and use a chess board colouring to construct for each of them a surface which has as its boundary this knot. What can you say about the orientability of these surfaces?



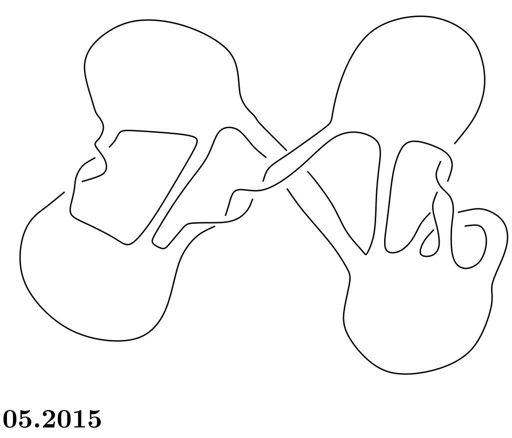


15. Use a chess board colouring (white outside) to construct two surfaces whose boundaries are the following links and then identify these surfaces.





16. Determine the genus of the following surface



**Due Date: 05.05.2015**