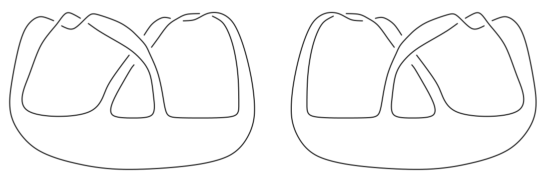
Exercisesheet 8: Seifert matrices and the Alexander polynomial

PART A: Seifert matrices

1. Complete the calculation of the Seifert matrix of the example in the lecture.

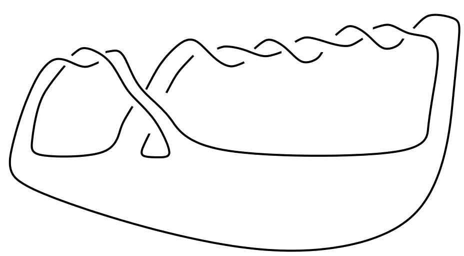
2. Consider the following Seifert surfaces for the trefoil and its mirror image.



   (a) Calculate in each case the Seifert matrix.

   (b) What is in general the relation between the Seifert matrix of a knot and of its mirror image?

3. Consider the following Seifert surface. Determine its Seifert matrix.



4. What is the effect of changing the orientation of the knot on the Seifert matrix?

5. Show that the following changes in the generators of the fundamental group of the Seifert surface result in Seifert equivalent Seifert matrices:

   (a) Changing the orientation on one of the α_i

   (b) Interchanging two curves α_i and α_j

   (c) Replacing a curve α_i by $\alpha_i + \alpha_j$.

6. Let K_1 and K_2 be two knots with Seifert surfaces F_1 and F_2. Consider the connected sum $K_1 \# K_2$ with Seifert surface $F_1 \# F_2$. How are the corresponding Seifert matrices related?

7. Show that the genus of a Seifert surface does not change under Reidemeister move I and therefore the Seifert matrix remains the same (i.e. no new curves!).

8. Let K be an oriented knot.

   (a) Denote by rK the same knot with the opposite orientation. Show that
   $$M_{rK} \overset{S}{\sim} M_K^T$$
   Here M^T denotes the transpose of the matrix M.

   (b) Let $\overline{K}$ be its mirror image. Show that
   $$M_{\overline{K}} \overset{S}{\sim} -M_K$$

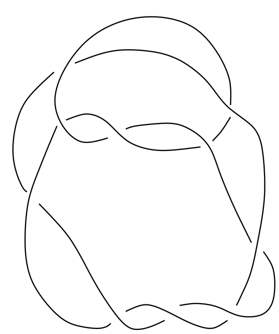
9. Find an example which shows that the determinant of a Seifert matrix is not a knot invariant.

10. Define the determinant of a knot by
   $$\det(K) := |\det(M_K + M_K^T)|$$

   (a) Show that this is a knot invariant.

   (b) Calculate $\det(K)$ for the unknot and the trefoil and show - once again - that the trefoil is non-trivial (define the determinant of the empty matrix to be one).

   (c) Calculate the determinant of the following knot. What can you now say about this invariant?



PART B: The Alexander polynomial

11. Calculate the Alexander polynomial for the knot from question 1.

12. Calculate the Alexander polynomial for the knot from question 3.

13. Let K be an oriented knot. Prove the following properties of the Alexander polynomial:

    (a) $\Delta_{rK}(t) = \Delta_K(t)$

    (b) $\Delta_{\overline{K}}(t) = \Delta_K(t)$

    And note that the Alexander polynomial is in this sense much weaker than the Jones polynomial which can detect mirror images.

14. Find a formula to calculate the Alexander polynomial for the connected sum of two knots $K_1 \# K_2$.

15. Prove the following statement: Suppose that K is a knot of genus $g(K)$, then the maximum degree of t in the Alexander polynomial cannot exceed $g(K)$.

16. Compute the Alexander polynomial of the figure eight knot using the skein relation.

17. Use the skein relation to prove that

$$\Delta_K(1) = 1$$

    for every knot K.