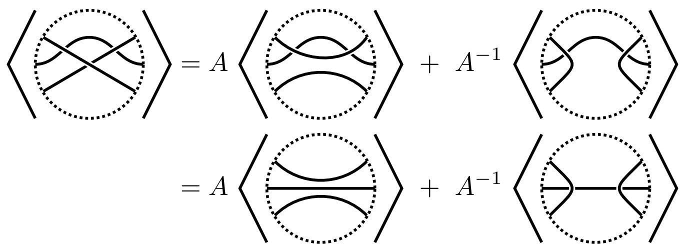
Solutions to sheet 4

Solution to exercise 1:

We have seen in the lecture that the Kauffman bracket is invariant under Reidemeister move 2. In particular, we have chosen the values in the skein relation accordingly (remember the knot algebra we did). With this, we can easily show the invariance of Reidemeister move 3 by:



Note that in the last expression above, both diagrams can be rotated by 180 degree without changing them and hence, this shows invariance under Reidemeister move 3.

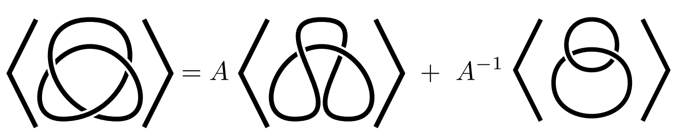
Solution to exercise 2:

We have the rules $\langle \bigcirc \rangle = 1$ and $\langle L \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$, so that

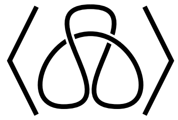
$$\begin{aligned}
\langle \sqcup^n \bigcirc \rangle &= (-A^2 - A^{-2})\langle \sqcup^{n-1} \bigcirc \rangle \\
&= (-A^2 - A^{-2})^2 \langle \sqcup^{n-2} \bigcirc \rangle \\
&= \dots \\
&= (-A^2 - A^{-2})^{n-1} \langle \bigcirc \rangle \\
&= (-A^2 - A^{-2})^{n-1}
\end{aligned}$$

Solution to exercise 3:

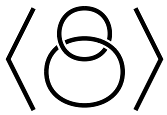
We start with the skein relation for one crossing of the trefoil, which gives:



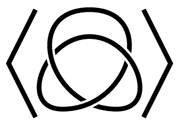
Recall that the Kauffman bracket is not invariant under Reidemeister move 1. More precisely, we have to multiply the Kauffman bracket by $-A^3$ when performing the Reidemeister move 1. Hence, the first bracket above is just $(-A^3)(-A^3)$ times the bracket of the unknot, which is 1 by definition. So, we have

A  $= A^7.$

Moreover, we have already seen in the lecture that the Kauffman bracket of the Hopf link is given by $-A^4 - A^{-4}$, hence

A^{-1}  $= -A^3 - A^{-5}.$

The Kauffman bracket of the trefoil is then

 $= A^7 - A^3 - A^{-5}.$

Solution to exercise 4:

(a) We have to show that the writhe is invariant under Reidemeister moves two (RII) and three (RIII). Let's start with RII: For any orientation of the strands we have two crossings with opposite signs before the move (i.e. that does not contribute to the writhe), and no crossing after the move. The writhe is therefore unchanged under RII. For RIII: For any orientation of the strands, the two crossings of the horizontal strand are replaced by two crossings with the same signs. The middle crossing is not affected, so the writhe is unchanged.

(b) Let D be a link diagram with c crossings and denote by w the writhe of D. Let c_1 be the number of positive crossings (i.e. those which contribute $+1$ to the writhe) and c_2 the number of negative crossings (i.e. those which contribute -1 to the writhe). We have $w = c_1 - c_2$ and $c = c_1 + c_2$. Combining those two equations give

$$c - w = (c_1 + c_2) - (c_1 - c_2) = 2c_2,$$

which means that c and w are either both even or both odd.

Solution to exercise 5:

Let T (trefoil) and H (Hopf link) be the two diagrams. Let T' be the diagram with reversed orientation and let H', H'' be the diagrams with one resp. two reversed components. We have:

$$w(T) = w(T') = -3,\ w(H) = w(H'') = 2,\ w(H') = -2$$

From problem 3 we have the result $\langle T\rangle = A^7 - A^3 - A^{-5}$ as well as the intermediate result $\langle H\rangle = -A^4 - A^{-4}$ (from the lecture). We obtain the following X-polynomials

$$\begin{aligned} X(T) = X(T') &= (-A)^9(A^7 - A^3 - A^{-5}) = -A^{16} + A^{12} + A^4 \\ X(H) = X(H'') &= (-A)^{-6}(-A^4 - A^{-4}) = -A^{-2} - A^{-10} \\ X(H') &= (-A)^6(-A^4 - A^{-4}) = -A^{10} - A^2. \end{aligned}$$

Solution to exercise 6:

Using the result of the previous problem we have

$$\begin{aligned} V(3_1) &= X(3_1)\big|_{A=t^{-\frac{1}{4}}} \\ &= -A^{16} + A^{12} + A^4\big|_{A=t^{-\frac{1}{4}}} \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

Changing the orientation has no influence on $V(3_1)$ by the previous problem.

Solution to exercise 7:

Let c be the number of crossings in a given diagram.

(a) The writhe is minimal if all crossings are negative and maximal if all crossings are positive. Hence, $-c \leq \sum s \leq c$.

(b) Same proof as in Exercise 4b).

Solution to exercise 8:

If D is an oriented knot diagram with reverse D' then $w(D) = w(D')$, since at every crossing, both directions change, i.e. every right-handed crossings stays right-handed and every left-handed crossing stays left-handed. Since the Kauffman bracket does not depend on orientations we have $V(D) = V(D')$.

Solution to exercise 9:

The state sum formula is

$$\langle D\rangle = \sum_s A^{\sum s} P^{|sD|-1}$$

where $P := -A^2 - A^{-2}$. Let's make the convention that a diagram without crossings has a single state s such that $sD = D$ and $\sum s$=0. (Recall that $|sD|$ is the number of disjoint loops.)

The above formula holds true when D is a diagram without crossings, i.e. the trivial link with n components (as seen in exercise 2). Now we can prove the formula for general diagrams by induction on the number of crossings.

Let D be a diagram which we resolve at the i-th crossing so that $\langle D\rangle = A\langle D_+\rangle + A^{-1}\langle D_-\rangle$. By induction hypothesis we have the following equalities:

$$\begin{aligned} A\langle D_+\rangle &= A \cdot \sum_{s \text{ state of } D_+} A^{\sum s} P^{|sD_+|-1} \\ &= \sum_{s \text{ state of } D_+} A^{\sum s+1} P^{|sD_+|-1} \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a positive crossing}}} A^{\sum s} P^{|sD|-1} \end{aligned}$$

Similar, we have

$$\begin{aligned} A^{-1}\langle D_+\rangle &= A^{-1} \cdot \sum_{s \text{ state of } D_-} A^{\sum s} P^{|sD_-|-1} \\ &= \sum_{s \text{ state of } D_-} A^{\sum s-1} P^{|sD_-|-1} \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a negative crossing}}} A^{\sum s} P^{|sD|-1} \end{aligned}$$

And hence

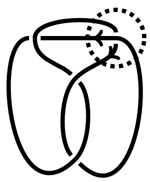
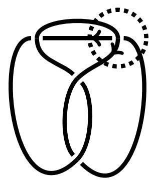
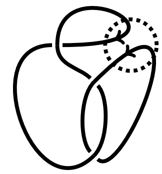
$$\begin{aligned} \langle D\rangle &= A\langle D_+\rangle + A^{-1}\langle D_-\rangle \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a positive crossing}}} A^{\sum s} P^{|sD|-1} + \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a negative crossing}}} A^{\sum s} P^{|sD|-1} \\ &= \sum_{s \text{ state of } D} A^{\sum s} P^{|sD|-1} \end{aligned}$$

which means that the state sum formula holds true for the diagram D.

Solution to exercise 10:

- The Kauffman bracket of the figure-eight-knot is given by $A^{-8} - A^{-4} + 1 - A^4 + A^8$ and the writhe is zero. Substituting $A \to t^{-\frac{1}{4}}$ gives the Jones polynomial
$$V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$$
- We use the skein relation
$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)$$

where L_+, L_- and L_0 are chosen as

$L_+ = 4_1$:  $L-$:  L_0 : 

We know that $V_{L_-} = 1$, as it is an unknot and $V_{L_0} = -t^{5/2} - t^{1/2}$ is the Hopf link (see exercise 11a)). Hence

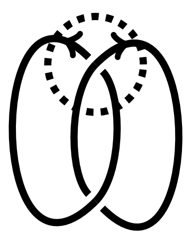
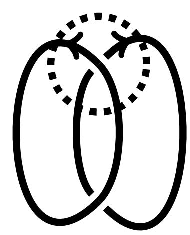
$$\begin{aligned} V(L_+) &= t^2 V(L_-) + (t^{3/2} - t^{1/2})V(L_0) \\ &= t^{-2} - t^{-1} + 1 - t + t^2. \end{aligned}$$

Solution to exercise 11:

(a) We use the skein relation

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)$$

to calculate the Jones polynomial of the Hopf link L. We use

$L_+ = L$:  $L-$: 

and L_0 is the unknot. L_- is the unlink with two components, i.e. we have $V_{L-}(t) = (-t^{1/2} - t^{-1/2})$ (for any orientation of the components). So we obtain

$$V_{L_+} = -t^{5/2} - t^{1/2}$$

Since this polynomial is not invariant under $t \mapsto t^{-1}$ we deduce that the Hopf link is not equivalent to its mirror image.

(b) Similarly, we can view the trefoil as L_+ and resolve one crossing to obtain L_-, which is the unknot and L_0, which is the negative Hopf link. We get:

$$\begin{aligned} V(L_+) &= t^2 V(L_-) + (t^{3/2} - t^{1/2})V(L_0) \\ &= t + t^3 - t^4. \end{aligned}$$

Solution to exercise 12:

(a) Let D_1, D_2 be oriented diagrams for K_1, K_2. Observe that every state s for $D := D_1 \# D_2$ corresponds to a pair (s_1, s_2), where s_i is a state for D_i. We write $s = (s_1, s_2)$. We have $\sum s = \sum s_1 + \sum s_2$ and $|sD| = |s_1 D_1| + |s_2 D_2| - 1$. We use these facts for the following computation, where $P = -A^2 - A^{-2}$:

$$\begin{aligned}
\langle D_1 \# D_2 \rangle &= \sum_s A^{\sum s} P^{|sD|-1} \\
&= \sum_{(s_1,s_2)} A^{\sum s_1 + \sum s_2} P^{|s_1 D| + |s_2 D| - 2} \\
&= \sum_{(s_1,s_2)} A^{\sum s_1} P^{|s_1 D|-1} A^{\sum s_2} P^{|s_2 D|-1} \\
&= \left(\sum_{s_1} A^{\sum s_1} P^{|s_1 D|-1} \right) \left(\sum_{s_2} A^{\sum s_2} P^{|s_2 D|-1} \right) \\
&= \langle D_1 \rangle \langle D_2 \rangle.
\end{aligned}$$

Now the fact that the writhe satisfies $w(D_1 \# D_2) = w(D_1) + w(D_2)$ yields

$$\begin{aligned}
X(K_1 \# K_2) &= (-A)^{-3w(K_1 \# K_2)} \langle K_1 \# K_2 \rangle \\
&= (-A)^{-3w(K_1)} \langle K_1 \rangle (-A)^{-3w(K_2)} \langle K_2 \rangle \\
&= X(K_1) X(K_2)
\end{aligned}$$

and after the substitution $A \mapsto t^{-1/4}$ we have the desired identity.

(b) The argument of part (a) needs only a slight modification in order to work here as well. Namely, for $s = (s_1, s_2)$ we now have $|sD| = |sD_1| + |sD_2|$. The result of the above computation then has an additional factor $P = -A^2 - A^{-2} = -t^{1/2} - t^{-1/2}$. Alternatively, one can apply the skein relation to a knot L for which $L_+ = L_- = K_1 \# K_2$ and $L_0 = K_1 \sqcup K_2$.

Solution to exercise 13:

Choose an orientation for the diagram. We apply the skein relation to the crossing below the top crossing. This is a left-hand crossing, hence the given diagram is L_+. One checks that L_- is the left-hand trefoil knot and L_0 is a negative Hopf link. Using results from previous problems we obtain

$$\begin{aligned}
V(L_-) &= t^{-2} V(L_+) + (t^{-3/2} - t^{-1/2}) V(L_0) \\
&= t^{-2}(-t^{-4} + t^{-3} + t^{-1}) + (t^{-3/2} - t^{-1/2})(-t^{-5/2} - t^{-1/2}) \\
&= -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}.
\end{aligned}$$