Solutions to sheet 5

Solution to exercise 1:

(a) Suppose L_+ has μ components. Then L_- has μ components as well, since no crossing is removed or added, only one over-crossing turns into an under-crossing or vice versa. For L_0, we have two possibilities, depending on how we connect the strings when removing a crossing. Either we connect two components of L_+ or we separate one component of L_+. Hence, L_0 has either $\mu+1$ or $\mu-1$ components.

(b) We proceed by induction on the complexity of a knot diagram D. Recall that the complexity was given by a pair (c,m), where c denotes the number of crossings in D and m the number of crossings needed to change D into a diagram of the unknot. The order is lexicographical. So, let us start with a diagram of complexity $(0,0)$, i.e. the trivial link with μ components. For $\mu=1$, we have a diagram of the unknot and hence $V(unknot)=1$, which has only integral powers in t and t^{-1}. We have seen on the last exercise sheet that

$$V(\cup^{\mu}\ unknot) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\mu-1}$$

which is a polynomial with integral powers in t and t^{-1} if μ is odd and a polynomial with half-integral powers in t and t^{-1} if μ is even. That shows the induction start.

For the induction step, recall the following Lemma from the lecture: *"Given an oriented link L with complexity (c,m), there exists a diagram D and a choice of crossing C so that the three links L_+, L_- and L_0 associated to D and C are L together with two new links of complexity strictly lower than (c,m)".*

Let us consider a diagram D with complexity (c,m). Let L_+ be the diagram corresponding to D and L_- and L_0 both of complexity lower than (c,m), by the lemma above (The other cases, i.e. for L_- corresponding to D or L_0 corresponding to D, can be treated similarly). Recall the Jones skein relation

$$t^{-1}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0),$$

which can be rewritten as

$$V(L_+) = t^2 V(L_-) + (t^{\frac{3}{2}} - t^{\frac{1}{2}})V(L_0). \tag{1}$$

Now, assume that the number of components μ of L_+ is even. By a), we know that the number of components of L_- is μ as well, hence even, and the number of components of L_0 is $\mu+1$ or $\mu-1$, hence odd. By the induction hypothesis, $V(L_-)$ consists of half integral powers of t and t^{-1} only and $V(L_0)$ consists of integral powers of t and t^{-1} only. Hence by (1), $V(L_+)$ consists of half integral powers of t and t^{-1} only.

Analogously for μ odd, which finishes the proof.

Solution to exercise 2:

Take an alternating diagram and insert two consecutive under-crossings by performing a Reidemeister move II anywhere in the diagram.

Solution to exercise 3:

That problem is still open.

Solution to exercise 4:

Recall that the Kauffman bracket is invariant under Reidemeister move 2 and 3. Only for Reidemeister move 1, we had to multiply the whole bracket by $-A^3$. Since multiplying the whole polynomial by $-A^3$ does not change the span (which is the difference between the highest and the lowest power), the span of the Kauffman bracket is actually invariant under Reidemeister 1 as well and hence a knot invariant.

Solution to exercise 5:

Let D be a reduced alternating diagram for K. From the lecture we know that the diagram is minimal, i.e. $c(D) = c(K)$. Moreover, we know that $\mathcal{B}(\langle D \rangle) = 4c(D)$. So we have

$$\mathcal{B}(V(K)) = \frac{1}{4}\mathcal{B}(X(K)) = \frac{1}{4}\mathcal{B}(\langle D \rangle) = \frac{1}{4} \cdot 4c(D) = c(D) = c(K).$$

Solution to exercise 6:

We claim that the knot $K_1 \# K_2$ is alternating. To see this, take alternating diagrams D_1 and D_2 for K_1 and K_2. Put them next to each other. There are two ways to obtain the connected some, as one diagram can be 'flipped' (i.e. turned 180 degrees). Both diagrams represent the connected sum $K_1 \# K_2$ and exactly one of them is alternating. Now we have, by the previous exercise:

$$\begin{aligned}
c(K_1 \# K_2) &= \mathcal{B}(V(K_1 \# K_2)) \\
&= \mathcal{B}(V(K_1)V(K_2)) \\
&= \mathcal{B}(V(K_1)) + \mathcal{B}(V(K_2)) \\
&= c(K_1) + c(K_2).
\end{aligned}$$

Solution to exercise 7:

Take any n-crossing diagram of the unknot, i.e. perform for example n Reidemeister moves 1. by Exercise 4, the span of the Kauffman bracket is a knot invariant and hence, it is zero for this diagram, since the span of the Kauffman bracket of the unknot is zero as well.

Solution to exercise 8:

(a) Assume that K is equivalent to its mirror image. This implies that the Jones polynomial $V(K)$ is invariant under the involution $t \mapsto t^{-1}$, hence it has the form

$$V(K) = \alpha_0 + \alpha_1(t + t^{-1}) + \cdots + \alpha_k(t^k + t^{-k})$$

with $\alpha_k \neq 0$. In particular, we have that $n = c(K) = \mathcal{B}(V(K)) = k - (-k) = 2k$ is even, which contradicts our assumption.

(b) We have $V(K\#K) = V(K)^2$, so the Jones polynomials have the form

$$V(K) = \alpha_l t^l + \cdots + \alpha_k t^k$$

and

$$V(K\#K) = \alpha_l^2 t^{2l} + \cdots + \alpha_k^2 t^{2k}$$

Now assume that $K\#K$ is equivalent to its mirror image. Since $V(K\#K)$ is invariant under $t \mapsto t^{-1}$ we have $-2l = 2k$, i.e. $-l = k$ and hence the span of $V(K)$ is equal to $2k$, so K must have even crossing number as we saw above. This means that $K\#K$ cannot be equivalent to its mirror image if K has odd crossing number.

Solution to exercise 9:

By problem 4 we have $\mathcal{B}(V(8_i)) = 8$ if the knot 8_i is alternating. For $i \in \{19, 20, 21\}$ the span of the Jones polynomial is different from 8 (which one can check in a knot table), and hence these knots are not alternating.

Solution to exercise 10:

Let H^+ be the positive Hopf link, i.e. the one with linking number $+1$. Consider L_+, L_-, L_0 such that $L_+ = H^+$. Then L_- is the unlink with two components and L_0 is the unknot. We already know that $P(L_0) = 1$, so what is left to do is to calculate $P(L_-)$.

For this, we consider three diagrams S_0, S_+ and S_- such that $S_0 = L_-$. Since S_0 is the unlink with two components, both S_+ and S_- are equal to the unknot. The skein relation says

$$l \cdot P(S_+) + l^{-1} \cdot P(S_-) = -m \cdot P(S_0)$$

which implies $P(L_-) = P(S_0) = -m^{-1}(l + l^{-1})$. Applying the skein relation again for the Hopf link, we get

$$P(H^+) = m^{-1}(l^{-1} + l^{-3}) - ml^{-1}.$$

We will also need the HOMFLY polynomial for the negative Hopf link H^-, which is

$$P(H^-) = m^{-1}(l + l^3) - ml.$$

Solution to exercise 11:

Let K be the left-hand trefoil and let K_+, K_-, K_0 such that $K_- = K$. Then we have K_+ the unknot and K_0 the negative Hopf link. Hence the skein relation reads

$$l \cdot P(unknot) + l^{-1} \cdot P(3_1) + m \cdot P(H^-) = 0$$

which, using the result from the previous problem, yields

$$P(K) = -2l^2 - l^4 + m^2 l^2.$$

Solution to exercise 12:

This follows from the fact that a crossing in an oriented knot K is positive/negative if and only if the same crossing is positive/negative for the reverse knot rK. This means that the skein relation looks the same for K and rK, so $P(K) = P(rK)$.

Solution to exercise 13:

(a) The skein relation says that

$$l \cdot P(L^+) + l^{-1} \cdot P(L^-) + m \cdot P(L_0) = 0.$$

where we take $L_0 = L \sqcup O$ and it turns out that $L_+ = L_- = L$, as Reidemeister move 1 (and its reversion) removes the part coming from the unknot in L_+ and L_-. Hence, we have

$$P(L \sqcup O) = -m^{-1}(l + l^{-1})P(L).$$

(b) We deduce this statement from (c). We have $L^+ = L^- = L_1 \# L_2$ and $L_0 = L_1 \cup L_2$, so applying (c) gives $P(L^+) = P(L^-) = P(L_1)P(L_2)$. The skein relation implies

$$P(L_1 \sqcup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2).$$

(c) We prove the statement by induction on the crossing number $c(L_2)$. If $c(L_2) = 0$ then L_2 is the unknot and $L_1 \# L_2 = L_1$, so the statement holds true since $P(O) = 1$. Now assume that $c(L_2) = n > 0$. Take a minimal diagram for L_2. Since it has an unknotting sequence we can pick a crossing such that the crossing number gets smaller if we change over- and undercrossing. Assume that there is a positive crossing with this property. We write $L_2^+ = L_2$ and consider the usual local variations L_2^- and L_2^0 at the crossing under consideration. We have $c(L_2^-), c(L_2^0) \leq n-1$, so by induction hypothesis

$$P(L_1 \# L_2^-) = P(L_1)P(L_2^-), \qquad P(L_1 \# L_2^0) = P(L_1)P(L_2^-)$$

and hence the skein relation

$$l \cdot P(L_1 \# L_2^+) + l^{-1} \cdot P(L_1 \# L_2^-) + m \cdot P(L_1 \# L_2^0) = 0$$

implies

$$\begin{aligned} l \cdot P(L_1 \# L_2) &= l \cdot P(L_1 \# L_2^+) \\ &= -l^{-1} \cdot P(L_1 \# L_2^-) - m \cdot P(L_1 \# L_2^0) \\ &= P(L_1)(-l^{-1} \cdot P(L_2^-) - m \cdot P(L_2^0)) \\ &= l \cdot P(L_1)P(L_2^+) \\ &= l \cdot P(L_1)P(L_2) \end{aligned}$$

which implies the statement. If we can't find an unknotting sequence for L_2 which begins with a positive crossing then we pick a negative crossing and set $L_2^- = L_2$ so that we can apply the same argument.

Solution to exercise 14:

We have two possible ways of connecting L_1 with L_2, namely (i) we can connect L_2 with the trefoil component of L_1 or (ii) with the trivial component of L_1. These two possibilities yield non-equivalent links $R_{(i)} = L_1 \#_{(i)} L_2$, $R_{(ii)} = L_1 \#_{(ii)} L_2$. However, by part (c) of the previous problem we have $P(R_{(i)}) = P(R_{(ii)}) = P(L_1)P(L_2)$.

Solution to exercise 15:

There is a mistake in the indicated substitution for m, the signs of the exponents are wrong. The right substitution is

$$l = it^{-1}, \qquad m = i(t^{-1/2} - t^{1/2}).$$

For an oriented link L let $P'(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ be the polynomial obtained by performing this substitution in the HOMFLY polynomial $P(L)$. The skein relation for P says

$$it^{-1}P'(L_+) - itP'(L_-) + i(t^{-1/2} - t^{1/2})P'(L_0) = 0,$$

i.e.

$$t^{-1}P'(L_+) - tP'(L_-) + (t^{-1/2} - t^{1/2})P'(L_0) = 0.$$

Furthermore we have $P'(O) = P(O) = 1$. We have seen that this skein relation together with the normalization on the unknot defines the Jones polynomial V uniquely, i.e. we have $P' = V$.

Solution to exercise 16:

In class we proved that for a two-component link L we have $a_1(L) = \mathrm{lk}(L)$. We assume here that L_+, L_- have one component, so L_0 has two components and therefore $a_1(L_0) = \mathrm{lk}(L_0)$. The skein relation for the Conway polynomial says that

$$z\nabla_{L_0}(z) = \nabla_{L_+}(z) - \nabla_{L_-}(z)$$

so by comparison of coefficients we get

$$\mathrm{lk}(L_0) = a_1(L_0) = a_2(L_+) - a_2(L_-).$$

Solution to exercise 17:

From Exercise 11 we know that the HOMFLY-polynomial of 3_1 is given by:

$$P(3_1) = -2l^2 - l^4 + m^2 l^2$$

The Alexander polynomial is then obtained by taking $l = i$ and $m = i(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, where $i = \sqrt{-1}$, so we get:

$$\Delta_{3_1}(t) = 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$$

The Conway polynomial is obtained by taking $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, i.e. we get

$$\nabla_{3_1}(z) = 1 + z^2$$

Similarly, the HOMFLY-polynomial of $\overline{3_1}$ is given by

$$P(3_1) = -2l^{-2} - l^{-4} + m^2 l^{-2}$$

which leads to

$$\nabla_{\overline{3_1}}(z) = 1 + z^2$$

as well. That means that the Conway polynomial cannot distinguish the trefoil knot from its mirror image.