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Numerical Analysis II

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D-MATH

Problem Sheet 8

Problem 8.1 Autonomisation and Strang-Splitting

The differential equation

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}, \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \quad \mathbf{A}(t) = t\mathbf{A}_0, \quad \mathbf{A}_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{8.1.1}$$

can be autonomised by introducing a variable $\mathbf{z} := \begin{bmatrix} \mathbf{y} \\ t \end{bmatrix}$, which yields the ODE

$$\dot{\mathbf{z}} = F(\mathbf{z}) = \begin{bmatrix} \mathbf{A}(t)\mathbf{y} \\ 1 \end{bmatrix}. \tag{8.1.2}$$

(8.1a) Formulate the discrete evolution of a Strang splitting method for (8.1.2) corresponding to the decomposition

$$F(\mathbf{z}) = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:\mathbf{f}(\mathbf{z})} + \underbrace{\begin{bmatrix} \mathbf{A}(t)\mathbf{y} \\ 0 \end{bmatrix}}_{=:\mathbf{g}(\mathbf{z})}$$

based on the exact evolutions for $\mathbf{f}$ and $\mathbf{g}$.

HINT: Use the representation of the exact solution of an IVP for a homogeneous linear differential equation .

(8.1b) Implement the resulting method by completing the MATLAB template

```
[t, y] = strangaut(y0, t0, T, h)
```

HINT: The matrix exponential function can be computed using `expm`.

(8.1c) Plot the numerical solution obtained by applying the splitting method to the IVP (8.1.1) where $\mathbf{y}_0 = [1, 0]^\top, t_0 = 0, T = 10, h = 0.01$. To do so, complete the template `strangautplot.m`. Use time t as your z-coordinate.

HINT: You might find the function `plot3` to be of use.

Problem 8.2 Strang-Splitting

For continuous differentiable function $\mathbf{g} : D \subset \mathbb{R}^d \mapsto \mathbb{R}^d$ consider the autonomous differential equation

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) \quad \text{with} \quad \mathbf{y} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{f}(\mathbf{y}) = \mathbf{f}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} := \begin{pmatrix} \mathbf{g}(\mathbf{v}) \\ \mathbf{g}(\mathbf{u}) \end{pmatrix}. \tag{8.2.1}$$

(8.2a) Give the method equations of the Strang-splitting 1-step method [NUMODE, Eq. (2.5.3)] for (8.2.1) which is based on the decomposition

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}_2(\mathbf{y}) + \mathbf{f}_1(\mathbf{y}) := \begin{pmatrix} 0 \\ \mathbf{g}(\mathbf{u}) \end{pmatrix} + \begin{pmatrix} \mathbf{g}(\mathbf{v}) \\ 0 \end{pmatrix}. \tag{8.2.2}$$

(8.2b) Is the 1-step method from subproblem (8.2a) an explicit or an implicit method?

(8.2c) What is the minimal order of this method in the general case?

(8.2d) Implement a MATLAB-function

```
u = sped(g,u0,T,N),
```

that realizes N equidistant iterations of the Strang-splitting 1-step method for (8.2.1) with initial value $\binom{\mathbf{u}_0}{\mathbf{u}_0}$ on the time interval $[0, T]$ using a *minimal number of* $\mathbf{g}(\cdot)$ *evaluations*. The function argument `g` is a handle `@(u) ....` The return value is a $d \times (N + 1)$ matrix of the values of the $\mathbf{u}$-component of the approximation $\mathbf{y}_k$, $k = 0, \dots, N$.

(8.2e) Write a MATLAB-function `spedcirc.m` that numerically solves the equation of motion for circular motion

$$\dot{x} = -y, \quad \dot{y} = x$$

for $x(0) = 1$, $y(0) = 0$ using the method `sped` from subproblem (8.2d) with $T = 60$, $N = 200$. It should also plot the solution.

(8.2f) Investigate numerically if the approximations $\mathbf{u}_k$ from `sped` in subproblem (8.2e) are on a circle.

Problem 8.3 Stability of a Decomposed Method

For the autonomous ODE $\dot{\mathbf{y}} = \mathbf{F}(\mathbf{y})$ we will consider the following decomposition of the right hand side $\mathbf{f}$, which we assume to be sufficiently smooth:

$$\mathbf{F}(\mathbf{y}) = \underbrace{\mathrm{D}\mathbf{F}(\mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*)}_{=:\mathbf{f}(\mathbf{y})} + \underbrace{\mathbf{F}(\mathbf{y}) - \mathrm{D}\mathbf{F}(\mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*)}_{=:\mathbf{g}(\mathbf{y})}. \tag{8.3.1}$$

Taking $\mathbf{y}^* = \mathbf{y}_0$ we define the discrete evolution $\mathbf{y}_1 = \boldsymbol{\Psi}^h \mathbf{y}_0$ of a single-step method as follows:

- Compute $\widetilde{\mathbf{y}}$ by performing one step of the explicit trapezoid rule with step-size $\frac{h}{2}$ applied to the initial value problem
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0.$$
- Compute $\widehat{\mathbf{y}}$ as the exact solution of the initial value problem
$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \qquad \mathbf{y}(0) = \widetilde{\mathbf{y}}$$
at the point $t = h$.
- Compute $\mathbf{y}_1$ by applying the explicit trapezoid rule with step-size $\frac{h}{2}$ to the initial vlaue problem
$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}), \qquad \mathbf{y}(0) = \widehat{\mathbf{y}}.$$

(8.3a) Determine the stability function of the method described above.

HINT: To check your results, use the MATLAB function `S=stabfn(z)` (it is given as a `pcode`). It evaluates the stability function at a given point z.

(8.3b) Complete the MATLAB template `plotstabdom` which plots the boundary of the stability domain. Save the resulting plot as `stabdom.eps`.

HINT: Use the MATLAB function `contour` to draw the boundary. Take -5 and 5 as bounds for both axes.

(8.3c) Show that the method described above has a convergence rate of at least 2.

HINT: Notice that the previously described method can be written as a well known (and well studied) method. A theorem from the lectures will then yield the statement.

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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