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Numerical Analysis II

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D-MATH

Problem Sheet 10

Problem 10.1 Evolution for Linearized ODE and Stability Function

Let $S(z)$ be the stability function ($\rightarrow$ [NUMODE, Thm. 3.1.6]) of a RK-SSM with the Butcher-Tableau

$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^\top \end{array}.$$

Furthermore, denote by $\boldsymbol{\Psi}_L^h$ the induced discrete evolution to the linearized ($\rightarrow$ [NUMODE, Thm. 3.2.8]) ODE

$$\dot{\mathbf{y}} = \mathrm{D}\mathbf{f}(\mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*).$$

Show that

$$\boldsymbol{\Psi}_L^h \mathbf{y}_0 = \mathbf{y}^* + S(h\mathrm{D}\mathbf{f}(\mathbf{y}^*))(\mathbf{y}_0 - \mathbf{y}^*).$$

HINT: cp. [NUMODE, Rem. 3.1.13]

Problem 10.2 Singly Diagonally Implicit Runge-Kutta Method

The scalar linear initial value problem of second order

$$\ddot{y} + \dot{y} + y = 0, \qquad y(0) = 1, \quad \dot{y}(0) = 0 \tag{10.2.1}$$

should be solved numerically using a 2-tep SDIRK-method (**S**ingly **D**iagonally **I**mplicit **R**unge-**K**utta Method). It is a Runge-Kutta method described by the Butcher-Tableau

$$\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1-\gamma & 1-2\gamma & \gamma \\ \hline & 1/2 & 1/2 \end{array}. \tag{10.2.2}$$

(10.2a) Calculate the equations for the increments k_1 and k_2 of the Runge-Kutta method (10.2.2) applied to the initial value problem corresponding to the differential equation $\dot{\boldsymbol{y}} = \mathbf{f}(t, \boldsymbol{y})$.

(10.2b) Show that, the stability function $S(z)$ of the SDIRK-method (10.2.2) is given by

$$S(z) = \frac{1 + z(1-2\gamma) + z^2(1/2 - 2\gamma + \gamma^2)}{(1-\gamma z)^2}$$

and plot the stability domain using the template `stabdomSDIRK.m`.

For $\gamma = 1$ is this method

- A–stable?

HINT: To investigate the A-stability, calculate the value of $S(z)$ on the imaginary axis $\mathrm{Re}\, z = 0$ and apply the maximum principle for holomorphic functions.

(10.2c) Formulate (10.2.1) as an initial value problem for a linear first order system for the function $\boldsymbol{z}(t) = (y(t), \dot{y}(t))^\top$.

(10.2d) Implement a MATLAB-function

```
z = sdirkStep(z0,h,gamma)
```

that realizes the numerical evolution of the 1-step method (10.2.2) for the differential equation determined in subproblem (10.2c).

(10.2e) Use MATLAB to conduct a numerical experiment, which gives an indication of the order of the method (with $\gamma = \frac{3+\sqrt{3}}{6}$) for the initial value problem from subproblem (10.2c). Choose $\boldsymbol{y}_0 = (1, 0)^\top$ as initial value, `T=10` as end time and `N=20,40,80,...,10240` as steps.

Problem 10.3

Assume that $\mathbf{A} \in \mathbb{C}^{d\times d}$ and $\sigma(\mathbf{A}) := \{\lambda : \ \lambda \text{ is an eigenvalue for } \mathbf{A}\} \subset \mathbb{C}^-$, where $\mathbb{C}^- := \{\mathbf{z} \in \mathbb{C} : \ \operatorname{Re}\mathbf{z} < 0\}$, and $\tau_* := \max\{\operatorname{Re}\lambda : \ \lambda \in \sigma(\mathbf{A})\}$. Prove that

$$\forall \beta \in (\tau_*, 0) \quad \exists C(\beta) \quad \text{such that} \quad \|\exp(tA)\| \leq C(\beta)e^{t\beta} \quad \forall t > 0.$$

HINT: Use the Jordan-Normal form of $\mathbf{A}$.

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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