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Numerical Analysis II

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D-MATH

Problem Sheet 11

Problem 11.1 Spectral Radius

Denote by $\mathbb{M}_n$ the set of all $n \times n$ matrices over $\mathbb{R}$ over $\mathbb{C}$. Let $\mathbf{A} \in \mathbb{M}_n$, and denote its spectrum by $\sigma(\mathbf{A})$. The spectral radius of $\mathbf{A}$ is defined as

$$\rho(\mathbf{A}) = \max_{\lambda \in \sigma(\mathbf{A})} |\lambda|.$$

We say that a matrix norm $\|\cdot\|$ on $\mathbb{M}_n$ is submultiplicative if

$$\|\mathbf{AB}\| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|,$$

holds for all $\mathbf{A}, \mathbf{B} \in \mathbb{M}_n$.

(11.1a) Show that $\rho(\cdot)$ is not a matrix norm.

HINT: Find a matrix norm axiom which the spectral norm does not satisfy.

(11.1b) Show that if $\|\cdot\|$ is a sub-multiplicative matrix norm, then for any matrix $\mathbf{A}$

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|.$$

HINT: Use sub-multiplicity to show that $|\lambda| \leq \|\mathbf{A}\|$ holds for all $\lambda \in \sigma(\mathbf{A})$.

(11.1c) Take an arbitrary $\mathbf{A} \in \mathbb{M}_n$. Prove the following

$$\rho(\mathbf{A}) = \inf\{\|\mathbf{A}\|, \|\cdot\| \text{ is a sub-multiplicative matrix norm }\}$$

HINT: One approach would be to show that every ϵ neighbourhood of $\rho(\mathbf{A})$ contains a matrix norm, i.e., there is a matrix norm $\|\cdot\|_*$ such that $\|\mathbf{A}\|_* \leq \rho(\mathbf{A}) + \epsilon$. Therefore, for each $\epsilon > 0$ one needs to construct a matrix norm satisfying such a condition. To construct a matrix norm it is recommended to use one of the various matrix decompositions based on eigenvalues (Schur, Jordan, etc.) and the fact that some well known induced matrix norms can be computed by means of a simple formula ($\|\cdot\|_1$, $\|\cdot\|_2$, etc.). You might also find it useful to show that if $\|\cdot\|$ is a (sub-multiplicative) matrix norm, then $\|\mathbf{B}^{-1}(\cdot)\mathbf{B}\|$ also defines a (sub-multiplicative) matrix norm.

(11.1d) Let $\mathbf{A}$ be an arbitrary element of $\mathbb{M}_n$, and let $\|\cdot\|$ be an arbitrary matrix norm. Show that

$$\rho(\mathbf{A}) = \lim_{k \to \infty} \|\mathbf{A}^k\|^{1/k}.$$

HINT: Bear in mind that the matrix norm used here is not necessarily sub-multiplicative. Hence, first show that the statement holds for sub-multiplicative norms and then use the equivalence of norms on finite dimensional vector spaces. It should follow clearly that $\rho(\mathbf{A}) \leq \lim_{k\to\infty} \|\mathbf{A}^k\|^{1/k}$. For the converse use (11.1c) and apply the norm equivalence.

Problem 11.2 Algebraic Stability

A Runge-Kutta 1-step method, represented by the Butcher-Tableau $\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^\top \end{array}$, is called *algebraically stable* provided its coefficients satisfy:

- $b_i \geq 0$, where $\mathbf{b} = (b_i)_{i=1}^s$,
- The matrix $\mathfrak{A}^\top \mathbf{D_b} + \mathbf{D_b}\mathfrak{A} - \mathbf{b}\mathbf{b}^\top$ is positive semi-definite.

where $\mathbf{D_b}$ denotes the diagonal matrix $\operatorname{diag}(b_1, \dots, b_s)$.

Algebraic stability is a sufficient condition for a discrete evolution of the Runge-Kutta method to inherit the non-expansivity (cf. [NUMODE, Def. 3.3.1] from the slides) of its corresponding continuous evolution. The advantage of this criterion is that it can be easily verified once the Butcher-Tableau is known.

(11.2a) Show that for a symmetric, positive semi-definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ and arbitrary vectors $\mathbf{u}_i \in \mathbb{R}^d,\ i = 1, \dots, n$, with $\mathbf{u}_i = (u_{ik})_{k=1}^d$ it holds that:

$$\sum_{i,j=1}^{n} m_{ij}\mathbf{u}_i^\top \mathbf{u}_j \geq 0, \quad \mathbf{M} = (m_{ij})_{i,j=1}^n$$

HINT: One way to show this is to use the fact that $\mathbf{M}$ can be orthogonally diagonalised by the principle axis transformation theorem. Furthermore, you might find the following useful

$$\text{if } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}, \text{ then } \sum_{i,j=1}^{n} a_{ij}b_{ij} = \operatorname{tr}(\mathbf{A}^\top \mathbf{B}) \tag{11.2.1}$$

(11.2b) Show that algebraic stability implies that the discrete evolution of a 1-step RK method, when applied to $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, is non-expansive if $\mathbf{f}$ is a dissipative vector field (see [NUMODE, Def. 3.3.5] in the slides) with respect to the Euclidean vector norm (i.e. $\mathbf{M} = \mathbf{I}$).

HINT: Use the step form of the increment equations

$$\mathbf{y}_1 = \mathbf{y}_0 + h\sum_{i=1}^{s} b_i \mathbf{f}(\mathbf{g}_i) \qquad \text{with} \qquad \mathbf{g}_i = \mathbf{y}_0 + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\mathbf{g}_j), \tag{11.2.2}$$

and the fact that the Euclidean vector norm is induced by the Euclidean scalar product, to rewrite $\left\|\mathbf{y}_1 - \hat{\mathbf{y}}_1\right\|^2$ as the sum of $\left\|\mathbf{y}_0 - \hat{\mathbf{y}}_0\right\|^2$ and two further terms. Here $\hat{\mathbf{y}}_1$ denotes the value that comes from applying one step of the RK to the initial value $\hat{\mathbf{y}}_0$.

Use (11.2.2) to replace $\mathbf{y}_0 - \hat{\mathbf{y}}_0$ in the second term. Then use dissipativity of the underlying vector field and subproblem (11.2a) to finish the proof.

Problem 11.3 A-Stable Non-Expansive Method

We know from the lectures that we can deduce the A-stability of a 1-step RK method from the fact that it inherits the non-expansiveness of an evolution. The following problem serves to show that the converse does not hold.

Consider the following autonomous differential equation

$$\dot{y} = f(y) = \begin{cases} -y^3 & y < 0, \\ -y^2 & y \geq 0. \end{cases} \tag{11.3.1}$$

(11.3a) Show that the continuous evolution of (11.3.1) is non-expansive.

(11.3b) Show that the implicit trapezoidal method

$$\mathbf{y}_1 = \mathbf{y}_0 + \frac{h}{2}\mathbf{f}(\mathbf{y}_0) + \frac{h}{2}\mathbf{f}(\mathbf{y}_1)$$

is an A-stable Runge-Kutta method.

(11.3c) Show that the implicit trapezoidal method does not inherit the non-expansivity of the continuous evolution of (11.3.1).

HINT: Find $y_0 < 0$ such that $|y_1| > |y_0|$. Why does this imply that the method is not non-expansive?

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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