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# Numerical Analysis II

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D-MATH

## Problem Sheet 12

### Problem 12.1 The Exponential Runge-Kutta Method

Consider the exponential Runge–Kutta single-step method

$$\begin{aligned}
\mathbf{k}_i &:= \phi(ch\mathbf{A})\big(f(\mathbf{u}_i) + h\mathbf{A}\sum_{j=1}^{i-1} c_{ij}\mathbf{k}_j\big), \quad i = 1,\dots,s, \\
\mathbf{u}_i &:= \mathbf{y}_0 + h\sum_{j=1}^{i-1} a_{ij}\mathbf{k}_j \quad i = 1,\dots,s, \\
\boldsymbol{\Psi}^h\mathbf{y}_0 &:= \mathbf{y}_0 + h\sum_{i=1}^{s} b_i\mathbf{k}_i.
\end{aligned} \tag{12.1.1}$$

for the autonomous differential equation $\dot{\mathbf{y}} = f(\mathbf{y})$, where $\mathbf{A} := Df(\mathbf{y}_0)$ and

$$\phi(z) = \frac{\exp(z) - 1}{z}.$$

**(12.1a)** Show that the two-step exponential Runge–Kutta method, with the parameter values

$$c = \frac{1}{2},\ a_{21} = \alpha,\ c_{21} = \frac{3}{4}\alpha^2 - \alpha,\ b_1 = 1 - \frac{1}{3\alpha^2},\ b_2 = \frac{1}{3\alpha^2}, \tag{12.1.2}$$

and $\alpha$ free to be chosen, solves the linear inhomogeneous differential equation $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{g}$, $\mathbf{A} \in \mathbb{R}^{d\times d}$, $\mathbf{g} \in \mathbb{R}^d$ exactly.

**(12.1b)** Implement a MATLAB function

```
function [t,y] = exponentialRK2(f,df,y0,T,h,alpha)
```

which solves the autonomous differential equation $\dot{\boldsymbol{y}} = f(\boldsymbol{y})$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$ with the 2-step Runge–Kutta method (12.1.1) and parameter values (12.1.2).

HINT: Even though the method (12.1.1) solves autonomous differential equations, write your implementation `exponentialRK2` for the general system $\dot{\boldsymbol{y}} = f(t, \boldsymbol{y})$ so it is compatible with MATLAB ODE-solvers.

**(12.1c)** Write a MATLAB function which determines the convergence rate of the method numerically. For your numerical experiment, solve Zeeman's heartbeat model

$$\begin{aligned} \dot{l} &= -l^3 + \alpha_{\text{heart}} l - p \\ \dot{p} &= \beta l \end{aligned} \tag{12.1.3}$$

with the parameter values $\alpha_{\text{heart}} = 3$, $\beta = 0.01$, initial values $(p(0), l(0)) = (0, 1)$ up to the end point $T = 100$. Use the step sizes $h = \frac{1}{2^k}$, $k = 2, \ldots, 6$, and as a reference solution, use the results given by `ode15s` with very high accuracy, f.e. `AbsTol=1e-12` and `RelTol=1e-12`.

## Problem 12.2 Robustness of L-Stable Method

In this exercise we investigate the *robustness* of L-stable 1-step methods for the scalar linear model problem [NUMODE, Eq. (3.1.1)]. Robustness denotes the special property of a numerical integrator, namely that the integration error can be reasonably estimated independently of a parameter in the differential equation.

We apply the two step Radau-2-method (→ [NUMODE, Eq. (3.6.4)]) to the model problem

$$\dot{y} = \lambda y, \qquad y(0) = 1. \tag{12.2.1}$$

Hence we construct an equidistant mesh with step size $h = \frac{1}{N}$ on the time interval $[0, 1]$.

**(12.2a)** Determine an analytic expression for

$$e_N(\lambda) := |y_N - y(1)|, \tag{12.2.2}$$

by expressing the numerical solution using the stability function (→ [NUMODE, Thm. 3.1.6]) of the method.

**(12.2b)** Write a MATLAB function

```
e = sup_e(N),
```

which determines a lower bound for

$$\sup_{0 \le \lambda \le N} e_N(\lambda)$$

by sampling on the interval $[0, N]$.

**(12.2c)** Give a constant $C$, independent of $h$ and $\lambda$, such that

$$|y_N - y(1)| \le Ch\,. \tag{12.2.3}$$

**(12.2d)** State a 1-step method which when applied to (12.2.1) with uniform time step size $h$ has the property (12.2.3) with $C$ independent of $h$ and $\lambda$.

## Problem 12.3 Exact Quadrature on $\mathcal{P}_{2s-2}$ Implies Positive Weights

It is often important to know whether the coefficients $b_i$ in the Butcher-scheme of a RK single step method are positive. Let the quadrature formula

$$\int_0^1 f(x)\,\mathrm{d}x \approx \sum_{i=1}^{s} b_i f(c_i) \quad c_i \neq c_j$$

be exact for polynomials $f \in \mathcal{P}_{2s-2}$. Show that the weights $b_i$ are necessarily positive.

HINT: Define to the $s$ different supporting points suitable polynomials $p_i(x) \in \mathcal{P}_{2s-2}$ with $p_i(x) \geq 0$.

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# References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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