Problem Sheet 2

Problem 2.1 Locally Area Preserving Evolution Operator

We will look at the Hamiltonian differential equation in $\mathbb{R}$:

$$\dot{p}(t) = -\frac{\partial H}{\partial q}(p(t), q(t)), \qquad \dot{q}(t) = \frac{\partial H}{\partial p}(p(t), q(t))\,,$$

where H is assumed to be twice continuously differentiable. The corresponding evolution operator [NUMODE, Def. 1.3.7]

$$\boldsymbol{\Phi}^t : \mathbb{R}^2 \to \mathbb{R}^2, \quad (p(0), q(0)) \mapsto (p(t), q(t))$$

defines a non-linear coordinate transformation for every point in time t. Show that this coordinate transformation is *area preserving*, in the sense that for a given area $A \subset \mathbb{R}^2$ we have:

$$\int_{\boldsymbol{\Phi}^t(A)} \mathrm{d}(p, q) = \int_A \mathrm{d}(u, v).$$

HINT: Look at the Jacobian determinant of the coordinate transformation as a function of time.

Problem 2.2 ODEs for Matrix-Valued Functions

Let the matrix-valued function $\mathbf{Y} : \mathbb{R} \to \mathbb{R}^{n \times n}$ satisfy the linear matrix differential equation

$$\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y} \quad \text{with} \quad \mathbf{A} \in \mathbb{R}^{n \times n}. \tag{2.2.1}$$

We have already met matrix differential equations in the form of the variational equation [NUMODE, Eq. (1.3.29)]. The simple single step methods discussed in [NUMODE, Sect. 1.3.3.4] carry over to matrix differential equations.

(2.2a) Compute $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{Y}^\top \mathbf{Y})$ and show that for skew-symmetric $\mathbf{A}$, i.e. $\mathbf{A} = -\mathbf{A}^\top$ we have:

$$\mathbf{Y}(0) \text{ orthogonal} \quad \Longrightarrow \quad \mathbf{Y}(t) \text{ orthogonal.}$$

(2.2b) Implement three MATLAB functions

(i) `function Y = ExplEulStep(A,Y0,h)`,

(ii) `function Y = ImplEulStep(A,Y0,h)`,

(iii) `function Y = ImplMidpStep(A,Y0,h)`,

which determine, for a given initial value $\mathbf{Y}(t_0) = \mathbf{Y}_0$ and for given step size h, approximations for $\mathbf{Y}(t_0 + h)$ using a step of

(i) the explicit Euler method [NUMODE, Sect. 1.4.1],

(ii) the implicit Euler method [NUMODE, Sect. 1.4.2],

(iii) the implicit mid-point method [NUMODE, Sect. 1.4.3].

(2.2c) Let $n = 2$, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{Y}(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Compute, using the three functions from subproblem (2.2b), discrete approximations $\mathbf{Y}_k$ of $\mathbf{Y}(kh)$, for $k = 1, \dots, 20$ with $h = 1/20$. Compare the norms $\left\|\mathbf{Y}_k^\top \mathbf{Y}_k - \mathbf{I}\right\|_F$, for $k = 1, \dots, 20$, and comment on your observations with regards to the result from subproblem (2.2a).

HINT: The Frobenius norm $\|\mathbf{A}\|_F$ of a matrix can be computed in MATLAB using the command `norm(A,'fro')`.

(2.2d) Show that the solution $\mathbf{Y}_k$ computed via the implicit mid-point rule satisfies:

$$\mathbf{Y}_0 \text{ orthogonal} \quad \Longrightarrow \quad \mathbf{Y}_k \text{ orthogonal.}$$

HINT: Show using the substitution $\mathbf{Y}_1 - \mathbf{Y}_0 = \frac{h}{2}\mathbf{A}(\mathbf{Y}_0 + \mathbf{Y}_1)$, that $2\mathbf{Y}_1^\top \mathbf{Y}_1 - 2\mathbf{Y}_0^\top \mathbf{Y}_0 = 0$.

Problem 2.3 Propagation Operator and Wronski Matrix

(2.3a) For the differential condition analysis of initial value problems with respect to perturbations of the initial data we have considered the differential of the solution and introduced it as the propagation matrix in [NUMODE, Sect. 1.3.3.4].

Prove the following property of the propagation matrix for all admissible arguments t, s

$$\mathbf{W}(t; s, \boldsymbol{\Phi}^{t,s}\mathbf{y}_0)^{-1} = \mathbf{W}(s; t, \mathbf{y}_0).$$

HINT: : [NUMODE, Sect. 1.3.3.4] and the property $\boldsymbol{\Phi}^{s,t} \circ \boldsymbol{\Phi}^{t,s}\mathbf{y} = \mathbf{y}$ of evolution operators.

(2.3b) Which of the three functions $\boldsymbol{\Phi}_i : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2,\ 1 \le i \le 3$ where

(i) $\boldsymbol{\Phi}_1(t, \mathbf{y}) := \boldsymbol{\Phi}_1^t \mathbf{y} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\mathbf{y}$,

(ii) $\boldsymbol{\Phi}_2(t, \mathbf{y}) := \boldsymbol{\Phi}_2^t \mathbf{y} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}\mathbf{y}$,

(iii) $\boldsymbol{\Phi}_3(t, \mathbf{y}) := \boldsymbol{\Phi}_3^t \mathbf{y} = \begin{pmatrix} \exp(\lambda t) & t \\ 0 & \exp(\lambda t) \end{pmatrix}\mathbf{y}, \quad \lambda \in \mathbb{R}$

satisfy the group property $\boldsymbol{\Phi}_i^{t+s} = \boldsymbol{\Phi}_i^s \circ \boldsymbol{\Phi}_i^t$ (see [NUMODE, Eq. (1.3.8)])? Which functions can be interpreted as evolution operators [NUMODE, Def. 1.3.7] of an autonomous differential equation $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, and which can't? Determine for the former ones the corresponding differential equations.

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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