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Spring Term 2015

Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 5

Problem 5.1 Composition of Runge–Kutta Methods

This exercise investigates the composition of Runge-Kutta methods. It turns out that such a composition is itself always another single step method of Runge-Kutta type:

Let $\hat{\Psi}^{t_0,t_0+h}$ and $\tilde{\Psi}^{t_0,t_0+h}$ be the discrete evolutions for two 2-stage Runge-Kutta single step methods defined by the Butcher-tableaux:

$$\begin{array}{c|cc} 0 & & \\ \widehat{c}_2 & \widehat{a}_{21} & \\ \hline & \widehat{b}_1 & \widehat{b}_2 \end{array} \quad \text{and} \quad \begin{array}{c|cc} 0 & & \\ \widetilde{c}_2 & \widetilde{a}_{21} & \\ \hline & \widetilde{b}_1 & \widetilde{b}_2 \end{array} \quad .$$

Show that their composition

$$\Psi^{t,t+2h} := \tilde{\Psi}^{t+h,t+2h} \circ \hat{\Psi}^{t,t+h}$$

can be interpreted as a discrete evolution of a Runge-Kutta method with step size $2h$, and determine the coefficients of this Runge-Kutta method.

Problem 5.2 Implicit Trapezoidal Rule

The implicit trapezoidal rule for the numerical solution of the scalar initial value problem $\dot{y} = f(t, y)$, $y(0) = y_0$ is given as

$$y_{k+1} = y_k + \frac{h}{2}(f_{k+1} + f_k),$$

where $f_k = f(t_k, y_k)$ and $h = t_{k+1} - t_k$.

(5.2a) The consistency error τ_k for the implicit trapezoidal rule along the solution trajectory $t \mapsto y(t)$ is given by

$$\tau_k = y(t_{k+1}) - (y(t_k) + \frac{h}{2}[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))]) \ .$$

Show that

$$\tau_k = -\frac{1}{12}h^3 \dddot{y}(\xi_k)$$

for some ξ_k in the interval (t_k, t_{k+1}).

(5.2b) Let f satisfy the global Lipschitz–condition

$$|f(t,u) - f(t,v)| \leq L|u - v| \,,$$

for all real t, u, v, where $L > 0$ is a constant independent of t. We further assume that $|\dddot{y}(t)| \leq M$ for all t and a constant $M > 0$.

Show that the global error $e_k = y(t_k) - y_k$ satisfies the inequality

$$|e_{k+1}| \leq |e_k| + \frac{1}{2}hL(|e_{k+1}| + |e_k|) + \frac{1}{12}h^3M.$$

For a constant step-size $h > 0$ with $hL < 2$, show that, if $y_0 = y(t_0)$, then

$$|e_k| \leq \frac{h^2M}{12L}\left[\left(\frac{1+\frac{1}{2}hL}{1-\frac{1}{2}hL}\right)^k - 1\right].$$

Problem 5.3 A MATLAB Function for Implicit Runge-Kutta Methods

Gauss collocation methods are in general implicit methods, therefore a non-linear system of equations must be solved in each step. To this end we use a damped Newton-method (`dampnewton.m`). The construction of collocation methods from [NUMODE, Sect. 2.2] of the lecture gives us a piecewise polynomial approximation $\mathbf{y}_h$ of the solution of the initial value problem, which is given by the collocation conditions on each interval of the time grid, as

$$\mathbf{y}_h(t_k + \tau h) = \mathbf{y}_0 + h\sum_{j=1}^{s}\mathbf{k}_j \int_0^{\tau} L_j(\xi)\,\mathrm{d}\xi, \quad 0 \leq \tau \leq 1,$$

with increments $\mathbf{k}_j$.

(5.3a) Extend the function `rkimplss.m`, such that it returns the approximations $\mathbf{y}_h(kh)$ for a given Butcher scheme, the time grid $\{0, h, 2h, \dots, 1\}$, $h = \frac{1}{N}$ and $k = 0, ..., N$ as well as the values $\mathbf{y}_h$ at the points $\{t_k + \tau_1 h, \dots, t_k + \tau_l h\}$, $0 < \tau_1 \leq \cdots \leq \tau_l = 1$.

HINT: Extend `collcoeffs.m`, such that for given $\tau_1, \dots, \tau_l$, the return value `b` has the components $b_{ji} = \int_0^{\tau_i} L_j(\xi)\,\mathrm{d}\xi$ and then modify `rkimplssm.m`.

(5.3b) Use the modified functions from subproblem (5.3a) and numerically show that an s-step Gauss collocation method on an equidistant timegrid need not adhere to the error estimate

$$\epsilon(h) := \max_{0\leq t\leq T}\|\mathbf{y}_h(t) - \mathbf{y}(t)\| = \mathcal{O}(h^{2s}).$$

To do so, complete the template `GaussCollLogRate.m` in which we estimate the convergence rate of a collocation method of order s, with $s = 1, \dots, 4$, by computing $\epsilon(h)$ on a grid with $N = 2^i$ points, where $i = 2, \dots, 6$ (i.e. as $h \to 0$).

HINT: As an example, use the scalar logistic differential equation

$$\dot{y} = 10y(1-y)$$

on $[0, 1]$ with $\mathbf{y}(0) = 0.01$.

(5.3c) Prove that the $\mathbf{y}_h$ computed with the 1-step Gauss collocation method satisfy

$$\max_{0\leq t\leq T} \|\mathbf{y}_h(t) - \mathbf{y}(t)\| = \mathcal{O}(h^2).$$

Published on 17 March 2015.

To be submitted by 24 March 2015.

References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

Last modified on March 16, 2015