P. Grohs
S. Hosseini
Ž. Kereta

Spring Term 2015

Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 6

Problem 6.1 Runge-Kutta Method of Consistency Order 3

Our aim in this exercise is to find sufficient and necessary conditions for a RK-one-step method of consistency order 3. As RK-one step methods are autonomisation-invariant [NUMODE, Rem. 2.3.15], we can limit our analysis to autonomous initial value problems, meaning

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{6.1.1}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function.

(6.1a) Find the Taylor series of $\boldsymbol{\Phi}^h \mathbf{y}_0$ around $h = 0$ up to $\mathcal{O}(h^4)$.

(6.1b) Find the taylor series around $h = 0$ of the discrete evolution given by a Runge-Kutta method [NUMODE, Def. 2.3.5]

$$\boldsymbol{\Psi}^h \mathbf{y}_0 = \mathbf{y}_0 + h \sum_{i=1}^{s} b_i \mathbf{k}_i$$

up to $\mathcal{O}(h^4)$.

HINT: The increments $\mathbf{k}_i$ are functions in h. It suffices to look at terms up to $\mathcal{O}(h^3)$.

(6.1c) Find the equation for the sufficient and necessary conditions for a RK-method of consistency order $p = 3$.

Problem 6.2 Simpson quadrature rule

The Simpson quadrature rule on the interval $[0, 1]$ uses the three sampling points $\tau_1 = t_0$, $\tau_2 = t_0 + \frac{1}{2}(t_1 - t_0)$ and $\tau_3 = t_1$ to approximate $\int_{t_0}^{t_1} f(\tau)\,\mathrm{d}\tau$. If we use these points as collocation points, we receive the so-called Simpson collocation method.

(6.2a) Compute the weights b_i, $i = 1, 2, 3$.

(6.2b) Compute the coefficients of the matrix $\mathbf{A}$ with $(\mathbf{A})_{ij} := a_{ij}$, $i, j = 1, 2, 3$.

(6.2c) Show that the Simpson collocation method has at a consistency order of at least 4.
HINT: Show that the Simpson quadrature rule is exact for $f \in \mathcal{P}_3$.

(6.2d) Verify that the Simpson collocation method for scalar autonomous differential equations is

$$y_{k+1} = y_k + \frac{h}{6}\Big(f(y_k) + 4f(y_{k+\frac{1}{2}}) + f(y_{k+1})\Big).$$

Problem 6.3 Property of Gauss Collocation

Let $\mathbf{V} : D \to \mathbb{R}$ be a differentiable function on the state space $D \subset \mathbb{R}^n$.

(6.3a) Characterise all right hand sides $\mathbf{f} : D \mapsto \mathbb{R}^n$ of the autonomous differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on the state space D for which the evolution $\boldsymbol{\Phi}$ satisfies the relationship

$$\mathbf{V}(\boldsymbol{\Phi}^t\mathbf{x}) \leq \mathbf{V}(\mathbf{x}) \tag{6.3.1}$$

for all $\mathbf{x} \in D$ and every admissible $t \in \mathbb{R}$.

(6.3b) Assume that $\mathbf{V} : D \to \mathbb{R}$ is a *quadratic* function and the evolution $\boldsymbol{\Phi}$ for the autonomous differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ satisfies the relationship $\mathbf{V}(\boldsymbol{\Phi}^t\mathbf{x}) \leq \mathbf{V}(\mathbf{x})$ for all $\mathbf{x} \in D$ and all admissible $t \in \mathbb{R}$. Show that, the Gauss collocation method, [NUMODE, Sect. 2.2.1], satisfies (6.3.1) for *quadratic* functions $\mathbf{V}$, i.e. for all $\mathbf{x} \in D$ and all admissible step sizes h

$$\mathbf{V}(\boldsymbol{\Psi}^h\mathbf{x}) \leq \mathbf{V}(\mathbf{x}).$$

HINT: Proof of [NUMODE, Thm. 3.3.7] from the lecture using subproblem (6.3a)

Published on 24 March 2015.

To be submitted by 31 March 2015.

References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

Last modified on March 26, 2015