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Problem Sheet 9

Problem 9.1 Stability Function of s-Stage Order s Explicit RK Method

Show that applying any *explicit* s-stage Runge-Kutta method of order s, with a stepsize h, to the model problem $\dot{y} = \lambda y$, where $\lambda \in \mathbb{C}$, yields

$$y_1 = \left(\sum_{j=0}^{s} \frac{z^j}{j!} \right) y_0, \quad z = h\lambda.$$

HINT: First show that y_1/y_0 is polynomial of degreee s in z and then use the Taylor expansion of the exact solution.

Problem 9.2 Stability of Extrapolation Methods

Extrapolation is a valuable technique for the construction of single-step methods of higher order. Of particular interest is the extrapolation of the explicit Euler method whose stability properties we will now study.

(9.2a) Consider the autonomous initial value problem

$$y' = \lambda y \qquad y(0) = 1, \qquad \lambda \in \mathbb{C}. \tag{9.2.1}$$

Let h be the base stepsize. Show that after $n \in \mathbb{N}$ steps of the explicit Euler with step size $h_n = \frac{h}{n}$ the approximate solution is of the form

$$y(h) \approx y_n = \left(1 + \frac{h\lambda}{n}\right)^n.$$

(9.2b) Show that the stability function $S(z)$ of the extrapolated Euler method after k steps with extrapolation sequence $\{n_i\}_{i=1}^k$ is given by the recursive formula

$$\begin{aligned} S_{i,1}(z) &= \left(1 + \frac{z}{n_i}\right)^{n_i} && \text{for } i = 1, \dots, k, \\ S_{i,\ell}(z) &= S_{i,\ell-1}(z) + \frac{S_{i,\ell-1}(z) - S_{i-1,\ell-1}(z)}{\frac{n_i}{n_{i-\ell+1}} - 1} && \text{for } 2 \leq \ell \leq i, \\ S(z) &:= S_{k,k}(z). \end{aligned}$$

HINT: You can use the Aitken-Neville scheme [NUMODE, Eq. (2.4.5)], [NUMODE, Eq. (2.4.6)]. Why?

(9.2c) Write the MATLAB function

```
function s = StabilityEval(z, n)
```

in which given `z` (a complex number) and a vector of integers `n` we compute `s`, the value of the stability function S, of the extrapolated Euler as defined in (9.2b), at the point `z`.

(9.2d) Complete the MATLAB function `StabilityDomain.m`, which plots the stability domain of the extrapolated Euler method using 5 steps extrapolation steps (i.e. `n=1:5`). Comment on the stability properties of this method.

(9.2e) Prove that the extrapolated Euler method is not A-stable for any extrapolation sequence.

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

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