Probabilities and statistics
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Serie 1

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Q1. We throw simultaneously two dices, one green and one red. Consider the following events:

- $W_1 :=$ Neither of the dices has a result greater than 2.
- $W_2 :=$ The green and the red one have the same number on them.
- $W_3 :=$ The number on the green is 3 times the number on the red.
- $W_4 :=$ The number on the red is by one greater than the number on the green one.
- $W_5 :=$ The number of the green one is greater or equal than the number on the red one.

(a) Write a suitable space Ω where all of these events can live.

(b) Describe W_i as a subsets of Ω.

(c) If you were colorblind (you cannot differentiate green and red). How does the sample space Ω change?, which W_i can live in this space?.

Q2. You have a box with $4k$ balls each one numerated with a different number in $\{1, ..., 4k\}$. At time j you take out one ball, look at its number and put it back, you repeat this experiments n times. Define

- $A_j :=$ The number taken out in the j-th time is bigger than $2k$.
- $B_j :=$ The number taken out in the j-th time is even.

(a) Write in terms of $(A_j)_{j=1}^n$ and $(B_j)_{j=1}^n$ the following events

i. $A :=$ Between 1 and n there was never a number bigger than $2k$.

ii. $B :=$ Between 1 and n there was at least one even number.

iii. $C :=$ The amount of balls bigger than $2k$ is bigger or equal than the amount of even balls.

(b) Describe in words the following events

i. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c$.

ii. $\bigcup_{j=1}^{n-2}(A_j \cap A_{j+1} \cap B_{j+2})$.

iii. $\bigcup_{m=1}^n \bigcap_{j=m}^n (A_j \cap B_j)$.

(c) For all $A \subseteq \Omega = \{1, .., 4k\}^n$ define

$$\mathbb{P}(A) = \frac{|A|}{(4k)^n}.$$

Show that for all strictly increasing sequences $(j_m)_{m=1}^{N_j}$, $(l_m)_{m=1}^{N_l}$ we have that

$$\mathbb{P}\left(\bigcap_{m=1}^{N_j} A_{j_m} \cap \bigcap_{m=1}^{N_l} B_{l_m}\right) = \left(\frac{1}{2}\right)^{N_l+N_j}.$$

Q3. Let $(A_j)_{j=1}^n$ be events:

(a) Show that:

$$\mathbf{1}_{\bigcup_{j=1}^n A_j} = 1 - \prod_{j=1}^n (1 - \mathbf{1}_{A_j}),$$

use it to prove that:

$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] = \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \mathbb{P}\left[\bigcap_{j=1}^k A_{i_j}\right]$$

(b) Using induction prove the following statements:

$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] \leq \sum_{j=1}^n \mathbb{P}[A_j] - \sum_{j=1}^{n-1} \mathbb{P}\left[A_j \cap A_{j+1}\right]$$
$$\mathbb{P}\left[\bigcup_{j=1}^n A_j\right] \geq \sum_{j=1}^n \mathbb{P}[A_j] - \sum_{i,j=1, i \neq j}^{n} \mathbb{P}\left[A_j \cap A_i\right]$$

Have a nice week ☺☻!!.