Probabilities and statistics

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Q1. Define $B_0 = \frac{1}{2}$. We want to play the following game: "In time n we bet $B_n = 2B_{n-1}$. Then we flip a coin X_n if it's -1 (Tail) we lose all the money we bet, if it's 1 (Head) we win the same amount of money we bet and we stop the game". Define $(V_n)_{n\in\mathbb{N}}$ the amount of money bet at time n:

(a) Compute the distribution of profit $(VX)_n$. What is the probability of losing?.

(b) Compute the expected value of $(VX)_n$. Then compute

$$\mathrm{Var}((VX)_n) := \mathbb{E}\left(((VX)_n - (\mathbb{E}(VX)_n))^2\right).$$

Q2. We are interested in studying the probability of success of a student at an entrance exam to two departments of a university. Consider the following events

$$\begin{aligned}
A &= \{\text{The student is man}\},\\
A^c &= \{\text{The student is woman}\},\\
B &= \{\text{The student applied for department I}\},\\
B^c &= \{\text{The student applied for department II}\},\\
C &= \{\text{The student was accepted}\},\\
C^c &= \{\text{The student wasn't accepted}\}.
\end{aligned}$$

We assume that we have the following probabilities (Berkeley 1973):

$\mathbb{P}(A) = 0.73,$
$\mathbb{P}(B \mid A) = 0.69, \quad \mathbb{P}(B \mid A^c) = 0.24,$
$\mathbb{P}(C \mid A \cap B) = 0.62, \quad \mathbb{P}(C \mid A^c \cap B) = 0.82, \quad \mathbb{P}(C \mid A \cap B^c) = 0.06, \quad \mathbb{P}(C \mid A^c \cap B^c) = 0.07.$

(a) Draw a tree describing the situation with the probabilities associated.

(b) Taking in to consideration the following probabilities $\mathbb{P}[C|A \cap B] = 0.62$, $\mathbb{P}[C|A^c \cap B] = 0.82$, $\mathbb{P}[C|A \cap B^c] = 0.06$, $\mathbb{P}[C|A^c \cap B^c] = 0.07$. With this information, do you think that in this examination women are disadvantaged?.

(c) Compute $\mathbb{P}(C \mid A)$ and $\mathbb{P}(C \mid A^c)$. Does this coincide with your answer of b)?.

Q3. Introduction to Bayesian Statistics We have m urns with red and white balls. The urn $i \in \{1, ..m\}$ has $2i-1$ red balls and $2m-2i+1$ white ones. We randomly select an urn and extract with replacement n times. Define:

$$X_j := \begin{cases} 1 & \text{If the } j\text{-th ball is red,} \\ 0 & \text{If the } j\text{-th ball is white.} \end{cases}$$

We are interested in the following problem " Given that you see $(X_j)_{j=1}^n$, can you say from which urn this balls were taken?".

(a) Compute $\mathbb{P}(X_1 = x_1, ..., X_n = x_n)$ for $x_i \in \{0, 1\}$ a possible result of the experiment. Are $X_1, .., X_n$ independent?.

(b) Compute the following probability

$$\mathbb{P}\left(\text{The box chose in } i \mid X_1 = x_1, ..., X_n = x_n\right).$$

Show that this only depends on the number or red balls, i.e., $k = \sum_{i=1}^{n} x_i$.

(c) Compute $\mathbb{P}\left(\text{The box chose in } i \mid X_1 = x_1, ..., X_n = x_n\right)$ for $m = 3$ and $n = 3$:

	$i = 1$	$i = 2$	$i = 3$
$k = 0$			
$k = 1$			
$k = 2$			
$k = 3$			

Q4. Monty hall problem. Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. j ($j = 2$ or $j = 3$), which has a goat. He then says to you, "Do you want to pick door No. l ($l = 2$ or $l = 3$ with $l \neq j$)?" Is it to your advantage to change your choice?

For trying to solve this problem we define the following

$$B_i = \text{"Auto is behind i."} \qquad (i = 1, 2, 3),$$
$$A_j = \text{"Moderator open door j."} \qquad (j = 2, 3).$$

(a) Define a natural model for the problem. Under this model compute $\mathbb{P}(B_i)$ for $i \in \{1, 2, 3\}$ and $\mathbb{P}(A_j|B_i)$ for $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$.

(b) Compute the Bayes formula $\mathbb{P}(B_1 \mid A_j)$ for $j \in \{2, 3\}$. Does opening the other door with a goat behind changes the probability that the car is behind door number 1 is?

(c) Would you change your choice?

Have a nice week ☺☻!!.