**Probabilities and statistics**

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# Serie 5

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**Q1.** Let $s \in (1, \infty)$. We define the Riemann's Zeta Function as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We want to prove that:

$$\zeta(s) = \frac{1}{\prod_{i=1}^{\infty}(1 - p_i^{-s})}.$$

where $p_1, p_2, .., p_k, .. = 2, 3, 5, 7, ....$ is the series of the ordered prime numbers.

**(a)** Take $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ with

$$\mathbb{P}(W) := \frac{1}{\zeta(s)} \sum_{n \in N} \frac{1}{n^s}, \quad W \in 2^{\mathbb{N}}.$$

Show that $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ is a probability space.

**(b)** Let $p$ be a prime number. Define $N_p := \{n \in \mathbb{N} : n \text{ is divisible by p.}\}$. Calculate $\mathbb{P}(N_p)$.

**(c)** Prove that the events $(N_p)_{p \text{ prime}}$ are independent under this probability measure.

**(d)** Compute

$$\mathbb{P}\left(\bigcap_{p \text{ prime}} N_p^c\right)$$

and conclude.

**Q2.** $\sigma$-ALGEBRAS

**(a)** Let $(\mathcal{A}_i)_{i \in I}$ be a family of $\sigma$-algebras. Show that $\bigcap_{i \in I} \mathcal{A}_i$ is a $\sigma$-Algebra.

**(b)** Prove that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\sigma$-algebras, $\mathcal{A}_1 \cup \mathcal{A}_2$ is a $\sigma$-Algebra iff $\mathcal{A}_1 \subseteq \mathcal{A}_2$ or $\mathcal{A}_2 \subseteq \mathcal{A}_3$.

**(c)** Let $\mathcal{A}$ be a $\sigma$-algebra and $\Psi$ an event. For $i \in \mathbb{N}$ define $A_i \in \mathcal{A}$ as " At time $i$ the event $\Psi$ occurs". Write, with the help of $A_i$ the following sets. Additionally show that they belong to $\mathcal{A}$.

i. " $\Psi$ never occurs"
ii. " $\Psi$ occurs infinitely many times".
iii. " From a moment onward $\Psi$ never occurs".
iv. " $\Psi$ occurs exactly twice".
v. " $\Psi$ occurs in total an odd number of times".

Which of them belong to the tail $\sigma$-algebra, i.e.,

$$\mathcal{A}_\infty := \bigcap_{n\in\mathbb{N}} \sigma(\{A_k : k \geq n\})?$$

**Q3.** Let $(\{0,2\}^{\mathbb{N}}, \mathcal{A}, \mathbb{P})$ be the model of infinite tossing of coins (Skript Satz 3.2, p. 40). We consider the random variable:

$$\begin{array}{rlcl} X: & \{0,2\}^{\mathbb{N}} & \longrightarrow & [0,1] \\ & \omega = (\omega_1, \omega_2, ...) & \mapsto & X(\omega) = \sum_{n=1}^{\infty} \frac{\omega_n}{3^n}. \end{array}$$

**(a)** Prove that $X$ is measurable.

**(b)** Show that the cumulative distribution function of $X$ is continuous.

**(c)** Prove that there exist disjoint intervals $I_k \subseteq [0,1]$ so that $F$ is constant in $I_k$ and $\lambda(\bigcup_{k=1}^{\infty} I_k) = 1$. (Where $\lambda$ is the Lebesgue measure)
**Hint:**

- $X(\omega) = \sum_{n=1}^{\infty} \frac{X_n(\omega)}{3^n}$.
- $F$ is constant on $X(\{0,2\}^n)^c$.

**Q4.** The Bertrand's paradox Consider an equilateral triangle inscribed in a circle of radius 1. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?. For solving this try the following probability models:

**(a)** The "random endpoints" method: Choose two uniform random points on the circumference of the circle and draw the chord joining them, i.e., let $U, V \sim U(0,1)$, define $X = e^{U2\pi i}, Y = e^{V2\pi i}$ and take the chord connecting $X$ and $Y$.

**(b)** The "random radius" method: Choose a radius of the circle, choose a uniform point on the radius and construct the chord through this point and perpendicular to the radius, i.e., choose a radius and choose $r \sim U(0,1)$, take the point on the radius that is at distance $r$ from the center and the chord will be the only chord perpendicular to this point.

**(c)** The "random midpoint" method: Choose a point uniformly anywhere within the circle and construct a chord with the chosen point as its midpoint, i.e., take $(x,y) \sim U(B(0,1))$ and take the chord whose midpoint is $(x,y)$.

**(d)** Is this a contradiction?

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