Probabilities and statistics

Lecturer: Prof. Dr. Sara van de Geer
Prof. Dr. Martin Larsson

Serie 8

April 28th, 2015

Q1. Let X_1 and X_2 follow a normal distribution with mean μ_i and variance σ_i^2. Prove that if X_1 is independent of X_2 then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Q2. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(Z_n)_{n\in\mathbb{N}}$ a sequence of random variables.

(a) Prove that if $Z_n \xrightarrow{\mathbb{P}} c \in \mathbb{R}$, then for all bounded and continuous functions f

$$\mathbb{E}\left(f(Z_n)\right) \to f(c).$$

(b) Show that if $Z_n \to c \in \mathbb{R}$ in distribution, then $Z_n \xrightarrow{\mathbb{P}} c$.

Q3. Take X_n i.i.d random variable so that

$$\mathbb{E}\left(X_1\right) = 1, \quad Var\left(X_1\right) = 2,$$

and define $S_n := \sum_{i=1}^n X_i$.

(a) Use Chebyshev-inequality to estimate

$$\mathbb{P}\left(\left|\frac{S_n}{n} - 1\right| \leq 0.5\right).$$

What is the value of the bound when $n = 40$.

(b) Use the Central Limit Theorem to estimate

$$\mathbb{P}\left(\left|\frac{S_n}{n} - 1\right| \leq 0.5\right).$$

What is the value of the bound when $n = 40$.

Q4. Take $x \in [0, 1]$. We say that x is normal if when we write x in its binary form,

$$x = \sum_{n\in\mathbb{N}} x_n 2^{-n} \qquad x_n \in \{0, 1\},$$

we have that $\lim_{n\to\infty} \frac{|\{1\leq k\leq n : x_k = 1\}|}{n} = \frac{1}{2}$.

(a) Prove that if we have a sequence $(U_n)_{n\in\mathbb{N}}$ i.i.d. Bernoulli with parameter $\frac{1}{2}$, then $U = \sum_{n\in\mathbb{N}} U_n 2^{-n}$ is an uniform random variable in $[0, 1]$

(b) Prove that if $U \sim U(0, 1)$, $\mathbb{P}(U \text{ is normal}) = 1$.