Probabilities and statistics

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Let Y be a random variable. We say that Y is infinitely divisible if for all $n \in \mathbb{N}$ there exists $(\xi_i^{(n)})_{i=1}^n$ i.i.d. so that

$$\sum_{i=1}^{n} \xi_i^{(n)} \stackrel{Dist}{=} Y.$$

Q1. We say that a random variable Y is bounded if there exists $M_Y \in \mathbb{R}$ such that $\mathbb{P}(|Y| \leq M_Y) = 1$. We want to understand bounded infinitely divisible random variables.

- **(a)** Prove that if Y is a constant (i.e. there exists $a \in \mathbb{R}$ so that $\mathbb{P}(X = a) = 1$) then Y is a bounded infinitely divisible random variable.
- **(b)** Suppose that Y is bounded and infinitely divisible. Take $\xi_i^{(n)}$ i.i.d. such that $\sum_{i=1}^n \xi_i^{(n)} = Y$, show that $\mathbb{P}(|\xi^{(n)}| \leq \frac{M_Y}{n}) = 1$.
- **(c)** Prove that Y is constant.
  Hint: Prove that $Var(Y) = 0$.

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Q2. Define $\phi_Y(\lambda) := \mathbb{E}\left(e^{i\lambda Y}\right)$ the characteristic function of Y. We want to understand the characteristic function of infinitely divisible random variables.

- **(a)** Prove that if $Y \sim N(\mu, \sigma^2)$ the Y is infinitely divisible.
- **(b)** Prove that Y is infinitely divisible iff for all n there exists $\phi_{n,Y}$, a characteristic function of a random variable, such that $(\phi_{n,Y}(\lambda))^n = \phi_Y(\lambda)$.
- **(c)** Prove that if Y is infinitely divisible and $\tilde{Y}$ is an independent copy of Y, then $X := Y - \tilde{Y}$ is infinitely divisible. Additionally, show that $0 \leq \phi_X(\lambda) \in \mathbb{R}$ for all $\lambda \in \mathbb{R}$.
- **(d)** Prove that $\phi_X(\lambda)^{\frac{1}{n}} = \phi_{n,X}(\lambda)$.
  Hint: It may be useful to prove that $0 \leq \phi_{n,X}(\lambda) \in \mathbb{R}$.
- **(e)** Prove that for all $\lambda \in \mathbb{R}$, $\phi_{n,X}(\lambda) \to \psi(\lambda)$ a function that is continuous in a neighborhood of 0.
- **(f)** Prove that for all $\lambda \in \mathbb{R}$, $\Phi_X(\lambda) \neq 0$. Conclude that $\Phi_Y(\lambda) \neq 0$.

Q3. In this question we want to use the criteria of the question one and two to see whether a random variable is infinitely divisible or not.

- **(a)** Let $Y \sim U(0,1)$ is it an infinitely divisible random variable?
- **(b)** Let $(\eta_i)_{i\in\mathbb{N}}$ a sequence of i.i.d random variables. Take $N \sim P(\varsigma)$ independent of $(\eta_i)_{i\in\mathbb{N}}$. Compute, in terms of ϕ_{η_1}, the characteristic function of $Y := \sum_{i=1}^N \eta_i$. Is it an infinitely divisible random variable?
  Remember that if $N \sim P(\varsigma)$

$$\mathbb{P}(N = k) = e^{-\varsigma}\frac{\varsigma^k}{k!} \quad k \in \mathbb{N}.$$