Probabilities and statistics
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Q1. Let X be a normal random variable.

(a) Prove that if we take $Y := X^2$, then $f_Y(y) = ce^{-\frac{y}{2}}\sqrt{y}\mathbf{1}_{\{y\geq 0\}}$ (We say that Y is distributed according to a χ-squared with one degree of freedom).

(b) If Y_1 and Y_2 are two independent copies of Y, prove that $f_{Y_1+Y_2} = c_2 e^{-\frac{x}{2}}\mathbf{1}_{\{x\geq 0\}}$. What is the name of this distribution.

(c) With the help of induction prove that $\sum_{i=1}^n Y_i$, where $(Y_i)_{i=1}^n$ are independent copies of Y, has as a density function

$$f_{\sum_{i=1}^n Y_i}(x) = c_n x \frac{n}{2} - 1 e^{-\frac{x}{2}}\mathbf{1}_{\{x\geq 0\}}.$$

This is call a χ-squared distribution with n degrees of freedom.

Q2. Take the following probability space $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda\mid_{[0,1]})$, where $\lambda\mid_{[0,1]}$ is the Lebesgue measure over $[0,1]$. Let $X_n(\omega) = \mathbf{1}_{A_n}(\omega)$ a sequence of random variables with $A_n \in \mathcal{B}([0,1])$.

(a) Under which condition for $(A_n)_{n\in\mathbb{N}}$ we have that $X_n \xrightarrow{\mathbb{P}} 0$.

(b) Write the event $\{\omega : X_n(\omega) \to 0\}$ with help of the sets $(A_n)_{n\in\mathbb{N}}$.

(c) Find a sequence $(A_n)_{n\in\mathbb{N}}$ of events so that $X_n \xrightarrow{\mathbb{P}} 0$ but $\{\omega : X_n(\omega) \to 0\} = \emptyset$.

Q3. Let $(X_i)_{i\geq 1}$ be a sequence of random variables with

$$\begin{aligned}
&\mathbb{E}(X_i) = \mu \quad \forall i,\\
&Var(X_i) = \sigma^2 < \infty \quad \forall i,\\
&Cov(X_i, X_j) = R(|i-j|) \quad \forall i,j.
\end{aligned}$$

Define $S_n := \sum_{i=1}^n X_i$.

(a) Prove that if $\lim_{k\to\infty} R(k) = 0$ then $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ in probability.

(b) Prove that if $\sum_{k\in\mathbb{N}} |R(k)| < \infty$ then $\lim_{n\to\infty} nVar(\frac{S_n}{n})$ exists.

Q4. Compute the limit of $\lim_{n\to\infty} e^{-n}\sum_{k=0}^n \frac{n^k}{k!}$
Hint: You can use the central limit theorem (Skript Theorem 4.3) for $(X_i)_{i\in\mathbb{N}}$ i.d.d. random variables such that $X_i \sim Poi(1)$.