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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 11

Problem 11.1 An Impossible Interpolation Estimate (Core problem)

[NPDE, Thm. 5.3.37] gave us bounds for the $L^2(\Omega)$-norm and $H^1(\Omega)$-seminorm of the error of piecewise linear interpolation on a triangular mesh of a bounded polygonal domain $\Omega \subset \mathbb{R}^2$. These bounds invariably contained the $H^2(\Omega)$-norm of the interpolated function. Now, somebody claims to have found an analogous interpolation estimate of the form

$$\|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \leq C h_{\mathcal{M}} \rho_{\mathcal{M}} |u|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{11.1.1}$$

with some constant $C > 0$.

(11.1a) Show that (11.1.1) implies

$$\|\mathsf{I}_1 u\|_{L^2(\Omega)} \leq C \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \tag{11.1.2}$$

with a constant $C > 0$ whose dependence of $h_{\mathcal{M}}$ and $\rho_{\mathcal{M}}$ should be made explicit.

HINT: First study [NPDE, Rem. 5.3.44].

Solution: Using the triangle inequality, we have:

$$\begin{aligned} \|\mathsf{I}_1 u\|_{L^2(\Omega)} &\leq \|u\|_{L^2(\Omega)} + \|u - \mathsf{I}_1 u\|_{L^2(\Omega)} \\ &\leq \|u\|_{L^2(\Omega)} + C_1 h_{\mathcal{M}} \rho_{\mathcal{M}} |u|_{H^1(\Omega)}, \end{aligned}$$

where we denoted by C_1 the constant in (11.1.1). Then we obtain (11.1.2) with $C = 1 + C_1 h_{\mathcal{M}} \rho_{\mathcal{M}} > 0$.

(11.1b) Argue why (11.1.2) cannot be true.

HINT: Remember [NPDE, Ex. 2.4.18], [NPDE, Cor. 2.4.24]. Note that we are in a 2D setting.

Solution: There is a function with finite H^1-norm which is unbounded in one point, see [NPDE, Ex. 2.4.18]. Hence the L^2-norm of its linear interpolant will be unbounded, if an interpolation node coincides with the location of the singularity.

Problem 11.2 Projection onto Constants (Core problem)

In [NPDE, Section 5.3.1] we derived L^2- and H^1-estimates for the error of piecewise linear interpolation on a grid, see [NPDE, Eq. (5.3.14)] and [NPDE, Eq. (5.3.16)]. The key tool was the integral representation formula [NPDE, Eq. (5.3.9)]. In this problem we practice these techniques for an even simpler projection operator.

Given a grid $\mathcal{M} := \{]x_{j-1}, x_j[\colon\ 1 \le j \le M\}$ of $[a,b] \subset \mathbb{R}$ we define a projection onto the space $\mathcal{S}_0^{-1}(\mathcal{M})$ of piecewise constant discontinuous functions on $\mathcal{M}$ according to

$$\mathsf{I}_0 : \begin{cases} L^2(]a,b[) & \mapsto \quad \mathcal{S}_0^{-1}(\mathcal{M}) \\ \quad u & \mapsto \quad \sum\limits_{j=1}^{M} \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(\xi)\ \mathrm{d}\xi \cdot \chi_{]x_{j-1},x_j[}, \end{cases} \tag{11.2.1}$$

where $\chi_{]x_{j-1},x_j[}$ stands for the characteristic function of the interval $]x_{j-1}, x_j[$, that is

$$\chi_{]x_{j-1},x_j[}(x) = \begin{cases} 1 & \text{, if } x \in]x_{j-1}, x_j[, \\ 0 & \text{else.} \end{cases} \tag{11.2.2}$$

We abbreviate $K :=]x_{j-1}, x_j[$ for some $j = 1, \dots, M$.

Remark: The linear projection I_0 is an instance of an L^2-projection. Generically, given a (closed) subspace $V \subset L^2(\Omega)$, the associated L^2-projection operator $\mathsf{Q}_V : L^2(\Omega) \mapsto V$ is defined through as solution operator of the variational problem

$$\mathsf{Q}_V u \in V : \quad \langle \mathsf{Q}_V u, v \rangle_{L^2(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} \quad \forall v \in V\ .$$

(11.2a) Compute $\mathsf{I}_0 u$ on $[0,1]$ for $u(x) = x$ and an equidistant mesh with meshwidth $h := M^{-1}$. Sketch the function $\mathsf{I}_0 u$.

Solution: We apply the definition of I_0 as from (11.2.1), and, since the interpolant is a staircase function, we obtain, by direct computation:

$$\begin{aligned} \mathsf{I}_0 u &= \sum_{j=1}^{M} \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(\xi)\, \mathrm{d}\xi \cdot \chi_{]x_{j-1},x_j[} \\ &= \sum_{j=1}^{M} \frac{1}{2} \frac{x_j^2 - x_{j-1}^2}{|x_j - x_{j-1}|} \cdot \chi_{]x_{j-1},x_j[} \\ &= \sum_{j=1}^{M} \frac{1}{2} \frac{(x_j + x_{j-1})(x_j - x_{j-1})}{|x_j - x_{j-1}|} \cdot \chi_{]x_{j-1},x_j[} \\ &= \sum_{j=1}^{M} \frac{(x_j + x_{j-1})}{2} \cdot \chi_{]x_{j-1},x_j[} \end{aligned}$$

Hence, $\mathsf{I}_0 u = \frac{(x_j + x_{j-1})}{2}$ in (x_{j-1}, x_j). The plot is given in Figure 11.1.

(11.2b) Derive the local integral representation formula for the projection error

$$(u - \mathsf{I}_0 u)(x) = \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \int_y^x u'(\xi)\ \mathrm{d}\xi\, \mathrm{d}y, \quad x_{j-1} < x < x_j. \tag{11.2.3}$$

HINT: Use the fundamental theorem of calculus [NPDE, Eq. (2.5.2)].

Solution: For $x_{j-1} \le x \le x_j$

$$(u - \mathsf{I}_0 u)(x) = u(x) - \mathsf{I}_0 u(x) = u(x) - \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(y)\, \mathrm{d}y$$

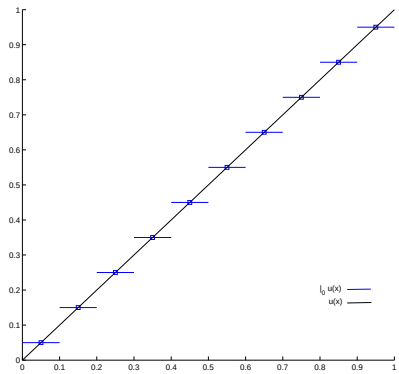

Figure 11.1: The function $u(x) = x$ and $\mathsf{I}_0 u$

Note that,

$$\int_{x_{j-1}}^{x_j} \int_y^x u'(\xi) d\xi \, \mathrm{d}y = \int_{x_{j-1}}^{x_j} (u(x) - u(y)) \, \mathrm{d}y = |x_j - x_{j-1}| u(x) - \int_{x_{j-1}}^{x_j} u(y) \, \mathrm{d}y$$

Then, using the definition of I_0 as from (11.2.1):

$$\begin{aligned}
(u - \mathsf{I}_0 u)(x) &= u(x) - \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(y) \, \mathrm{d}y \\
&= \frac{|x_j - x_{j-1}|}{|x_j - x_{j-1}|} u(x) - \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} u(y) \, \mathrm{d}y \\
&= \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \int_y^x u'(\xi) \, \mathrm{d}\xi \, \mathrm{d}y
\end{aligned}$$

(11.2c) Starting from (11.2.3) deduce the estimate

$$\|u - \mathsf{I}_0 u\|^2_{L^2(]x_{j-1}, x_j[)} \leq |x_j - x_{j-1}|^2 |u|^2_{H^1(]x_{j-1}, x_j[)}. \tag{11.2.4}$$

HINT: Apply the Cauchy-Schwarz inequality for integrals [NPDE, Eq. (2.3.15)] twice.

Solution:

$$
\begin{aligned}
\|u - \mathsf{I}_0 u\|^2_{L^2(]x_{j-1},x_j[)} &= \int_{x_{j-1}}^{x_j} \left| \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \int_y^x u'(\xi)\,\mathrm{d}\xi\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&= \left(\frac{1}{|x_j - x_{j-1}|} \right)^2 \int_{x_{j-1}}^{x_j} \left| \int_{x_{j-1}}^{x_j} \int_y^x 1 \cdot u'(\xi)\,\mathrm{d}\xi\,\mathrm{d}y \right|^2 \mathrm{d}x \\
&\leq \left(\frac{1}{|x_j - x_{j-1}|} \right)^2 \int_{x_{j-1}}^{x_j} \left[\left(\int_{x_{j-1}}^{x_j} 1^2\,\mathrm{d}y \right) \left(\int_{x_{j-1}}^{x_j} \left| \int_y^x u'(\xi)\,\mathrm{d}\xi \right|^2 \mathrm{d}y \right) \right] \mathrm{d}x \\
&= \left(\frac{1}{|x_j - x_{j-1}|} \right)^2 \int_{x_{j-1}}^{x_j} (x_j - x_{j-1}) \left(\int_{x_{j-1}}^{x_j} \left| \int_y^x u'(\xi)\,\mathrm{d}\xi \right|^2 \mathrm{d}y \right) \mathrm{d}x \\
&= \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \left(\int_{x_{j-1}}^{x_j} \left| \int_y^x 1 \cdot u'(\xi)\,\mathrm{d}\xi \right|^2 \mathrm{d}y \right) \mathrm{d}x \\
&\leq \frac{1}{|x_j - x_{j-1}|} \int_{x_{j-1}}^{x_j} \left(\int_{x_{j-1}}^{x_j} \left[\left(\int_{x_{j-1}}^{x_j} 1^2\,\mathrm{d}\xi \right) \left(\int_{x_{j-1}}^{x_j} |u'(\xi)|^2\,\mathrm{d}\xi \right) \right] \mathrm{d}y \right) \mathrm{d}x \\
&= \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} |u|^2_{H^1(]x_{j-1},x_j[)}\,\mathrm{d}y\,\mathrm{d}x \\
&= |x_j - x_{j-1}|^2 |u|^2_{H^1(]x_{j-1},x_j[)},
\end{aligned}
$$

where for the last inequality, while applying Cauchy-Schwarz, we also used the positivity of the integrand and the monotonicity of the interval to have $\int_y^x |u'(\xi)|^2\,\mathrm{d}\xi \leq \int_{x_{j-1}}^{x_j} |u'(\xi)|^2\,\mathrm{d}\xi = |u|^2_{H^1(]x_{j-1},x_j[)}$.

(11.2d) Based on (11.2.4) derive the global projection error estimate

$$
\|u - \mathsf{I}_0 u\|_{L^2(]a,b[)} \leq h_{\mathcal{M}} |u|_{H^1(]a,b[)}, \tag{11.2.5}
$$

where $h_{\mathcal{M}}$ is the meshwidth of $\mathcal{M}$.

Solution:

$$
\begin{aligned}
\|u - \mathsf{I}_0 u\|^2_{L^2(]a,b[)} &= \sum_{\mathcal{M}} \|u - \mathsf{I}_0 u\|^2_{L^2(]x_{j-1},x_j[)} \\
&\leq \sum_{j=1}^{M} |x_j - x_{j-1}|^2 |u|^2_{H^1(]x_{j-1},x_j[)} \\
&\leq h^2_{\mathcal{M}} |u|^2_{H^1(]a,b[)}
\end{aligned}
$$

Hence

$$
\|u - \mathsf{I}_0 u\|_{L^2(]a,b[)} \leq h_{\mathcal{M}} |u|_{H^1(]a,b[)}.
$$

Problem 11.3 Convergence of Finite Element Solutions (Core problem)

A student is testing his implementation of a finite element method. On the square domain $\Omega = (0,1)^2$ he considers the 2nd-order elliptic boundary value problem

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega, \\ u &= \frac{1}{4}\left(1 - \|\boldsymbol{x}\|^2\right) && \text{on } \partial\Omega. \end{aligned} \tag{11.3.1}$$

He computes an approximate solutions u_N by means of a finite element Galerkin method using linear (piecewise first order polynomials) and quadratic (piecewise second order polynomials) finite elements, denoted by LFE and QFE respectively, on a sequence of triangular meshes $\mathcal{M}$.

The following table lists the measured $H^1(\Omega)$-seminorm of the discretization error as a function of the meshwidth h.

h	0.70	0.35	0.17	0.088	0.044	0.022	0.011
LFE	0.10	0.051	0.025	0.012	0.0064	0.0032	0.0008
QFE	$1.75\cdot10^{-16}$	$1.24\cdot10^{-15}$	$5.71\cdot10^{-15}$	$2.29\cdot10^{-14}$	$8.91\cdot10^{-14}$	$3.53\cdot10^{-13}$	$1.41\cdot10^{-12}$

(11.3a) Show that $u(\boldsymbol{x}) = \frac{1}{4}(1 - \|\boldsymbol{x}\|^2)$ is the exact solution of (11.3.1)

Solution: We have

$$-\Delta u = -\frac{\mathrm{d}^2 u}{\mathrm{d}{x_1}^2} - \frac{\mathrm{d}^2 u}{\mathrm{d}{x_2}^2} = \frac{1}{2} + \frac{1}{2} = 1.$$

This solution also (obviously) matches the Dirichlet boundary data.

(11.3b) What kind of convergence (qualitative and quantitative) for linear Lagragian finite elements can be inferred from the error table?

Solution: When creating a log-log plot of the error norm versus the meshwidth $h_{\mathcal{M}}$ the data points are approximately located on a straight line. This hints at algebraic convergence. The rate is given by the slope of the line and can be read off the plot as ≈ 1. You may also use `polyfit` to determine the rate by linear regression as done in the following MATLAB code.

Listing 11.1: Convergence plot

```
data = load('cvgtab.mat');

p = polyfit(log(data.h), log(data.H1S_Error_LFE), 1);
p(1)
```

Alternatively, you may have noticed that the values of $h_{\mathcal{M}}$ roughly shrink by a factor of two when advancing to the next finer mesh. The same is true of the error. Hence the quotients

$$\texttt{error} : h_{\mathcal{M}}$$

are about constant, which, again, confirms first order convergence.

(11.3c) Explain the striking difference between the behavior of the discretization error for linear and quadratic Lagrangian finite elements.

Solution: For $p = 2$ the errors are around machine precision, because the exact solution u is a quadratic polynomial and belongs to the finite element space $\mathcal{S}_2^0(\mathcal{M})$. The errors reflect nothing more than numerical roundoff. For $p = 1$ no the solution has to be approximated and such "free lunch" exists.

Problem 11.4 Localized Interpolation Error Estimates

There is a more refined way than that of [NPDE, Thm. 5.3.37] to state interpolation error estimates. It relies on the piecewise constant *meshwidth function*

$$\hbar(\boldsymbol{x}) = h_K \quad \text{if} \quad \boldsymbol{x} \in K, \tag{11.4.1}$$

where K is a cell of a triangular mesh $\mathcal{M}$ of a domain $\Omega \subset \mathbb{R}^2$.

Based on [NPDE, Lemma 5.3.34] derive the estimate

$$\left\|\hbar^{-2}(u - \mathsf{I}_1 u)\right\|_{L^2(\Omega)} \leq C|u|_{H^2(\Omega)} \quad \forall u \in H^2(\Omega), \tag{11.4.2}$$

What is a concrete value for the constant C?

Solution: Using [NPDE, Lemma 5.3.34]:

$$\left\|\hbar^{-2}(u - \mathsf{I}_1 u)\right\|_{L^2(K)}^2 \leq \frac{3}{8} h_K^4 \left|\hbar^{-2} u\right|_{H^2(K)}^2 = \frac{3}{8}|u|_{H^2(K)}^2.$$

Then, summing over all the triangles:

$$\left\|\hbar^{-2}(u - \mathsf{I}_1 u)\right\|_{L^2(\Omega)}^2 \leq \sum_{K \in \mathcal{M}} \frac{3}{8}|u|_{H^2(K)}^2 = \frac{3}{8}|u|_{H^2(\Omega)}^2$$

$$\left\|\hbar^{-2}(u - \mathsf{I}_1 u)\right\|_{L^2(\Omega)} \leq \sqrt{\frac{3}{8}}|u|_{H^2(\Omega)}.$$

Problem 11.5 Shape Regularity and Angle Condition

When deriving interpolation error estimates for linear interpolation on triangular meshes in [NPDE, Section 5.3.2], it was convenient to introduce the concept of a shape regularity measure ρ_K for a triangle K, see [NPDE, Def. 5.3.36]. The intuition is that in 2D the shape regularity measure indicates the degree to which a triangle is distorted. This distortion was linked to the angles in [NPDE, Fig. 198], [NPDE, Fig. 199], and [NPDE, Fig. 200]. This link will be explored in this problem.

Bound the smallest angle of a triangle K by an expression involving only ρ_K.

HINT: From secondary school recall the formula $|K| = \frac{1}{2}ab\sin(\gamma)$, where γ is the angle enclosed by the sides with lengths a, b.

Solution: Assume, $\gamma \leq \beta \leq \alpha$ are related with edges $c < b < a$, respectively (refer to Figure 11.2). Then, we know by the triangle inequality that $b + c > a$, and so $2b > a$. Then,

$$\sin\gamma = \frac{2}{ab} \cdot \frac{1}{2}ab\sin\gamma < \frac{4}{a^2}|K| = \frac{4}{\rho_K}.$$

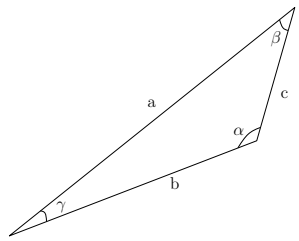

Figure 11.2: A Triangle

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References

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