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Numerical Methods for Partial Differential Equations

ETH Zürich
D-MATH

Homework Problem Sheet 4

Problem 4.1 Establishing empirical convergence rates

In [NPDE, Section 1.6] the concept of "convergence", and in particular of *algebraic* and *exponential* convergence, has been introduced. Through several practical examples you have learned, how use error norms obtained from numerical experiments to extract the necessary information to describe *qualitatively* and *quantitatively* the convergence of a numerical method. Detailed instructions are given in [NPDE, § 1.6.26]. This problem is meant to make you more familiar with concepts and techniques connected with the notion of "convergence".

(4.1a) In the supplementary material distributed with this problem sheet you can find the ASCII files `Nvalues.txt`, `Error1.txt`, `Error2.txt` and `Error3.txt`.

The files `Error1.txt`, `Error2.txt` and `Error3.txt` contain sequences of error norms obtained from three different numerical experiments and three different discretization schemes for some boundary value problem. For each case the associated values N of degrees of freedom used for discretization are contained in the file `Nvalues.txt`. Thus pairs of problem sizes N and related norms of the discretization error are available.

For each of the three convergence studies, describe qualitatively and quantitatively the empirical convergence that you observe.

HINT: Valuable information can be extracted from doubly logarithmic (`loglog`) or semi-logarithmic (`semilogy`) plots. In MATLAB, you may use the function `polyfit` to estimate the convergence rates. See [NPDE, § 1.6.26] for details.

HINT: The MATLAB function `lsqcurvefit` that solves nonlinear least squares problems may come into help.

(4.1b) For each of the three convergence studies from subproblem (4.3a), provide a plot of the error norms versus N, for which the measured error norms approximately lie on a straight line.

Problem 4.2 $L^2(0,1)$-Orthogonal Projection onto Linear Finite Element Space

In this problem we deal with a very simple *quadratic* minimization problem that does not even involve derivatives. We derive the associated variational formulation, and then discretize it with linear finite elements as in [NPDE, Section 1.5.2.2]. A careful re-examination of this section is recommended. You will be asked to implement the method in MATLAB and perform a numerical study of its convergence.

Given $f \in C^0_{\text{pw}}([0,1])$, the minimization problem reads:

$$u^* = \operatorname*{argmin}_{v \in C^1_{\text{pw}}([0,1])} \underbrace{\int_0^1 |v(x) - f(x)|^2 \,\mathrm{d}x}_{:=J(v)} \tag{4.2.1}$$

(4.2a) Show that $J = J(v)$ from (4.2.1) is a quadratic functional and identify its building blocks according to [NPDE, Def. 2.2.24].

(4.2b) Derive the variational problem associated with the minimization problem (4.2.1).

HINT: Don't forget to specify the trial and test spaces.

(4.2c) Show that the solutions to (4.2.1) are unique, that is, if two solutions are known to be global minimizers of J, then they most agree.

HINT: A theorem from [NPDE, Section 2.2.3] may come handy. If you plan to use it, please give a precise citation.

To begin with, we consider Galerkin discretization based on the space $V_N = \mathcal{S}^0_1(\mathcal{M})$ of piecewise linear continuous functions on a uniform mesh $\mathcal{M}$ of $[0,1]$ with M cells of size $h = \frac{1}{M}$, see [NPDE, Section 1.5.2.2]. In the following, use tent functions according to [NPDE, Eq. (1.5.62)] as a basis for the Galerkin trial and test space.

(4.2d) What is the Galerkin matrix for this problem? Write a MATLAB function

```
A = galmatrix_tent(M)
```

which takes the number of equal mesh cells as input in `M`, computes the Galerkin matrix for the variational problem from subproblem (4.2b), and returns it in the *sparse matrix* `A`.

(4.2e) Write down the entries of the right-hand side vector for the variational problem from subproblem (4.2b), using the composite trapezoidal rule for numerical quadrature of integral involving the generic function f.

(4.2f) Write a MATLAB function

```
L = rhs_tent(M,f)
```

which takes as input the number `M` of equal mesh cells and a function handle to f, computes the right-hand side vector, and returns it in the column vector `L`.

(4.2g) Write a MATLAB function

```
U = l2proj_tent(M,f)
```

that solves the variational problem from task (4.2b) approximately based on linear finite element Galerkin discretization on an equidistant mesh with `M` cells. The arguments `M` and `f` are the same as before. The column vector `U` should contain the value of the solution in each node of the mesh.

We now want to investigate the convergence for the L^p-*norm* of the discretization error.

For $u \in C^0_{\text{pw}}(I)$ on a closed interval $I \subset \mathbb{R}$ and a real number $1 \le p \le \infty$, the L^p-*norm* of u is defined as

$$\|u\|_{L^p(I)} := \left(\int_I |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}, \qquad \text{for } 1 \le p < \infty, \tag{4.2.2}$$

and

$$\|u\|_{L^\infty(I)} := \sup_{x \in I} |u(x)|. \tag{4.2.3}$$

(4.2h) Write a MATLAB function

```
function rate = lpcvg(sol,p)
```

that, given $1 \le p \le \infty$, performs a convergence study for the L^p-norm of the discretization error and returns in `rate` the (algebraic) convergence rate. Use the convention that $p = 0$ for the L^∞-norm. The argument `sol` is a function handle to the exact solution. Use the values $N = 10 \cdot 2^i$, $i = 1, \dots, 9$, for the number of mesh intervals.

For the computation of the norms, use the 2-point Gauss quadrature rule, with quadrature points $\zeta_1 = -\frac{\sqrt{3}}{3}, \zeta_2 = \frac{\sqrt{3}}{3}$ and weights $\omega_1 = \omega_2 = 1$ on the reference interval $[-1, 1]$.

(4.2i) Write a MATLAB script

```
convergence
```

to compute

$$\|u - u_N\|_{L^p(]0,1[)} \qquad \text{for } N \to \infty. \tag{4.2.4}$$

for the values $p = 1.0, 1.5, 2, 4, \infty$. In (4.2.4), u and u_N denote the exact and Galerkin solutions, respectively. Plot the error curves for these values of p in the case that the exact solution is $u(x) = \sin(x^2)$ and in the case that it is $u(x) = \begin{cases} 1 & x \le \frac{\sqrt{2}}{2} \\ 0 & x > \frac{\sqrt{2}}{2} \end{cases}$.

Listing 4.1: Testcalls for Problem 4.2

```
mesh = linspace(0,1,5)';
fprintf('\n\n##galmatrix_tent:')
full(galmatrix_tent(mesh))

fprintf('\n\n##rhs_tent:')
rhs_tent(mesh, @(x) x)'

fprintf('\n\n##l2proj_tent:')
l2proj_tent(mesh, @(x) x)'

fprintf('\n\n##lpcvg:')
rate = lpcvg(@(x) sin(x.^2),2)
```

Listing 4.2: Output for Testcalls for Problem 4.2

```
test_call_lp

##galmatrix_tent:
ans =

    0.0833    0.0417         0         0         0
    0.0417    0.1667    0.0417         0         0
         0    0.0417    0.1667    0.0417         0
         0         0    0.0417    0.1667    0.0417
         0         0         0    0.0417    0.0833

##rhs_tent:
ans =

         0    0.0625    0.1250    0.1875    0.1250

##l2proj_tent:
ans =

   -0.1429    0.2857    0.5000    0.7143    1.1429

##lpcvg:
rate =

    1.5141
```

Problem 4.3 Quadratic and non-quadratic functionals

In [NPDE, Section 1.3] you have seen many examples of energy functionals, and you have learned the connection between the minimization of a functional and the solution of a variational formulation. Moreover, in [NPDE, Section 1.4] and, again, in [NPDE, Section 2.2.3], you have studied *quadratic* minimization problems, which lead to *linear* variational formulations as explained in [NPDE, Section 1.4.1] and [NPDE, Section 2.4.1]. In this problem, we will study the properties of further functionals.

Let $V := \mathcal{C}^1_{\mathrm{pw}}([0,1])$. We consider the functionals $J_i : V \to \mathbb{R}$, $i = 1,2,3$:

$$J_1(v) := \int_0^1 |v(x)|^2 - v'(x)\,\mathrm{d}x\ , \tag{4.3.1}$$

$$J_2(v) := \int_0^1 v(x)v'(x) + v^2(x)\,\mathrm{d}x\ , \tag{4.3.2}$$

$$J_3(v) := \int_0^1 \cosh(v'(x)) + v(x)\,\mathrm{d}x\ , \tag{4.3.3}$$

$\cosh(x) := \frac{1}{2}(e^x + e^{-x})$.

(4.3a) Which J_i, $i = 1,2,3$, is a quadratic functional?

For these, identify the associated linear forms and (symmetric) bilinear forms. Which of the latter are symmetric positive definite?

(4.3b) Show that J_1 and J_3 are convex.

HINT: Recall the following definition:

A functional $J : V \to \mathbb{R}$ on an affine space V is called convex if

$$J(\lambda x + (1-\lambda)y) \leq \lambda J(x) + (1-\lambda)J(y), \qquad \text{for all } \lambda \in [0,1] \text{ and all } x, y \in V. \tag{4.3.4}$$

HINT: Use that $f(x) : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, and $\cosh : \mathbb{R} \to \mathbb{R}$ are convex on $V = \mathbb{R}$.

(4.3c) Derive the variational formulations that have to be satisfied by potential minimizers of J_i, $i = 1, 2, 3$.

HINT: For J_3, remember that $(\cosh(x))' = \sinh(x)$.

(4.3d) State the 2-point boundary value problems satisfied by solutions of the variational equations from subproblem (4.3c), when $V_0 = C^1_{\mathrm{pw},0}([0,1])$ is used as trial and test space. In all cases, assume the solution u to be smooth.

(4.3e) Show that no minimizer exists for J_1.

(4.3f) Show that no minimizer exists for J_2.

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References

[NPDE] Lecture Slides for the course "Numerical Methods for Partial Differential Equations".SVN revision # 74075.

[1] M. Struwe. Analysis für Informatiker. Lecture notes, ETH Zürich, 2009. https://moodle-app1.net.ethz.ch/lms/mod/resource/index.php?id=145.

[NCSE] Lecture Slides for the course "Numerical Methods for CSE".

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