10.1. Fact 1 about quotients Let $1 \leq p < \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. Define, for fixed $1 \leq i \leq n$ and $h \in \mathbb{R} \setminus \{0\}$ by

$$u^h(x) := \frac{u(x + he_i) - u(x)}{h}.$$

Prove that

$$\left\| u^h \right\|_{L^p(\mathbb{R}^n)} \leq \|\partial_i u\|_{L^p(\mathbb{R}^n)}.$$

Hint: Start with $u \in C^\infty(\mathbb{R}^n)$ and use the fundamental theorem of calculus.

10.2. Fact 2 about quotients For $1 < p < \infty$, $u \in L^p(\mathbb{R}^n)$, we define u^h as in 10.1. Furthermore, we assume that

$$\sup_{h>0} \left\| u^h \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

Prove that u has a weak derivative in the i-th direction in $L^p(\mathbb{R}^n)$.

Hint: Prove that $\int_{\mathbb{R}^n} \varphi u^h = \int_{\mathbb{R}^n} \varphi^{-h} u$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and combine this with Banach-Alaoglou.

10.3. Laplace on $\mathbb{R}^n$. The purpose of this exercise is to establish the similar estimate for Δ as in the lecture course for $\Omega = \mathbb{R}^n$.[1]. The main difference is that for working on $\mathbb{R}^n$, the expression $K_j * f$ makes only sense for compactly supported functions, thus we indicate steps in this exercise to circumvent these difficulties.

We want to prove the following, for all $n \in \mathbb{N}$, $1 < p < \infty$, there is $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \leq \sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} \langle \nabla \varphi, \nabla u \rangle}{\|\nabla \varphi\|_{L^q(\mathbb{R}^n)}}. \tag{1}$$

(a) Prove there is a unique bounded operator $T : L^p(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$Tf = \sum_{i=1}^n \nabla(K_i * f_i)$$

for all $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. **Hint:** Use Calderòn–Zygmund.

[1] Cf. the notes provided on the webpage.

(b) For $u \in C_0^\infty(\mathbb{R}^n)$, there exists $f \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \langle f, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle \nabla u, \nabla \varphi \rangle$$

where $\|f\|_{L^p(\mathbb{R}^n,\mathbb{R}^n)} = \sup_{0 \neq \varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\langle \nabla u, \nabla\varphi\rangle|}{\|\nabla\varphi\|_{L^q(\mathbb{R}^n)}}$. **Hint:** Use Hahn-Banach.

(c) With f, and u as in (b), prove that $Tf = \nabla u$.

Hint:

(i) Prove that $T\nabla\varphi = \nabla\varphi$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

(ii) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $h \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\int_{\mathbb{R}^n} \langle Tg, h \rangle = \int_{\mathbb{R}^n} \langle g, Th \rangle .$$

(iii) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, prove that $\int_{\mathbb{R}^n} \langle g, \nabla\varphi \rangle = \int_{\mathbb{R}^n} \langle Tg, \nabla\varphi \rangle$.

(iv) For $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, give a sequence $\varphi_\nu \in C_0^\infty(\mathbb{R}^n)$ with $\|Tg - \nabla\varphi_\nu\|_{L^p} \to 0$ as $\nu \to \infty$.

(v) Prove that $T^2 = T$.

(vi) Prove that for $g \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ the following are equivalent.

(α) $Tg = 0$

(β) $\int_{\mathbb{R}^n} \langle g, \nabla\varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

Hint: Prove for $(\beta) \Rightarrow (\alpha)$ that $\int_{\mathbb{R}^n} \langle T^2 g, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

(vii) Conclude.

(d) Prove (1).

10.4. Why -1? For $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ bounded and open, prove that the weak derivative $\partial_i u : C_0^\infty(\Omega) \to \mathbb{R}$ of a function $u \in L^p(\Omega)$ can be extended in a unique way to an element of $W^{-1,p}(\Omega)$. Thereby, we define a bounded linear operator $\partial_i : L^p(\Omega) \to W^{-1,p}(\Omega)$ and so the notation seems natural.

Please hand in your solutions for this sheet by Monday 09/05/2016.