14.1. $L^q - L^p$ **spaces.** Let $1 < p, q < \infty$.

(a) Define

$$A_{q,p} := \{u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R} : u \text{ is measurable}, \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^p \, \mathrm{d}x\right)^{\frac{q}{p}} \mathrm{d}t\right)^{\frac{1}{q}} < \infty\}/_\sim.$$

where $\sim$ is equivalence almost everywhere. Prove that $A_{q,p} \cong L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

Hint: Use strong measurability to get a sequence of step functions $u_i \in L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ and establish the correspondance. Use $v_k = \sum_{i=0}^k |u_{i+1} - u_i|$ with

$$\|u_{i+1} - u_i\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq \frac{1}{2^i}$$

and the norm on $L^1([-T,T], L^1(K))$ for $T > 0$ and $K \subset \mathbb{R}^n$ compact to prove that the limiting function is measurable.

(b) Prove that $C_0^\infty(\mathbb{R}\times\mathbb{R}^n) \subset L^q(\mathbb{R}, L^p(\mathbb{R}^n))$ is dense.

Hint: Reduce it to step functions and then smoothen every single value, and then smoothen in the t-variable.

(c) Prove that for $\beta_\delta(t,x) = \rho_\delta(t)\prod_{i=1}^n \rho_\delta(x)$. Prove that $\beta_\delta * u \to u$ in $L^q(\mathbb{R}, L^p(\mathbb{R}^n))$.

Hint: Banach-Steinhaus (2.1.5).

(d) Prove that $C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ is dense in $W^{1,q}(\mathbb{R}, L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}, W^{2,p}(\mathbb{R}^n))$.

14.2. Hardy's inequality [1] Fix $1 < p < \infty$ and $a > 0$.

(a) Let $f : (0,\infty) \to \mathbb{R}$ be a Lebesgue measurable and suppose that the function $(0,\infty) \to \mathbb{R} : x \to x^{p-1-a}|f(x)|^p$ is integrable. Show that the restriction of f to each interval $(0,x]$ is integrable and prove Hardy's inequality

$$\left(\int_0^\infty x^{-1-a}\left|\int_0^x f(t)\mathrm{d}t\right|^p \mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{a}\left(\int_0^\infty x^{p-1-a}|f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}}.$$

Show that equality in Hardy's inequality holds if and only if $f = 0$ almost everywhere.

Hint: Assume first that f is nonnegative with compact support and define

$$F(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \qquad \text{for } x > 0.$$

[1] This is part of exercise 4.52 p:171 in Dietmar's Measure and Integration book.

Use integration by parts to obtain

$$\int_0^\infty x^{p-1-a} F(x)^p \, \mathrm{d}x = \frac{p}{a} \int_0^\infty x^{p-1-a} F(x)^{p-1} f(x) \, \mathrm{d}x.$$

Use integration by part.

(b) Show that the constant $\frac{p}{a}$ in Hardy's inequality is sharp.

Hint: Choose $\lambda < 1 - \frac{a}{p}$ and take $f(x) = x^{-\lambda}$ for $x \leq 1$ and $f(x) := 0$ for $x > 1$.

(c) Let $f : (0, \infty) \to \mathbb{R}$ be Lebesgue measurable and $|f|^p$ be Lebesgue integrable. Prove that

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t) \, \mathrm{d}t \right|^p \, \mathrm{d}x \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p \, \mathrm{d}x.$$

(d) Same assumptions on f as in (a). Show that the restriction to each interval $[x, \infty)$ is integrable and prove the inequality

$$\left(\int_0^\infty x^{a-1} \left| \int_x^\infty f(t) \mathrm{d}t \right|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq \frac{p}{a} \left(\int_0^\infty x^{p-1-a} \, |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Hint: Apply the inequality in (a) to the function $g(x) := x^{-2} f(x^{-1})$.

14.3. Uniform maximal regularity of S and its dual.

Prove that for $1 < q, q^* < \infty$ and X a reflexive complex Banach space with $\frac{1}{q} + \frac{1}{q^*} = 1$, the following are equivalent.

- S is uniformly maximal q-regular.
- S^* is uniformly maximal q^*-regular.

where S^* is the dual strongly continuous semigroup (7.3.9).

Hint: Prove by passing to the limit that for $g : [0, T] \to X^*$, $f : [0, T] \to X$ C^1, we have

$$\int_0^T \left\langle g(t), A \int_0^t S(t-s) f(s) \, \mathrm{d}s \right\rangle \mathrm{d}t = \int_0^T \left\langle A^* \int_0^s S^*(s-t) g(T-t) \, \mathrm{d}t, f(T-s) \right\rangle \mathrm{d}s$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing between X and its dual.

14.4. Sobolev–Slobodeckij space Let $n \in \mathbb{N}$ and fix real numbers $p \geq 1$ and $0 < s < 1$. The completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm

$$\|f\|_{w^{s,p}} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \right)^{1/p}$$

is called the homogeneous Sobolev–Slobodeckij space and is denoted by $w^{s,p}(\mathbb{R}^n, \mathbb{C})$. On the other hand, the completion of $C_0^\infty(\mathbb{R}^n, \mathbb{C})$ with respect to the norm

$$\|f\|_{b^{s,p}_{p,1,int}} := \left(\int_0^\infty \frac{1}{r^{sp+1}} \frac{1}{\mu(B_r)} \int_{B_r} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, \mathrm{d}x \, \mathrm{d}h \, \mathrm{d}r\right)^{1/p}$$

is called the homogeneous Besov space and denoted by $b^{s,p}_p(\mathbb{R}^n, \mathbb{C})$.

Prove that $w^{s,p}(\mathbb{R}^n, \mathbb{C}) = b^{s,p}_p(\mathbb{R}^n, \mathbb{C})$.

Hint: Prove that $\|f\|_{b^{s,p}_{p,1,int}} = \left(\frac{1}{(n+sp)\mu(B_1)}\right)^{1/p} \|f\|_{w^{s,p}}$ by using Fubini and spherical coordiantes.

14.5. Besov for $1 \leq s < 2$**.** Fix $p, q \geq 1$. Prove that for $f \in C_0^\infty(\mathbb{R}^n)$ non constant and $1 \leq s < 2$, that

$$\|f\|_{b^{s,p}_{q,1}} = \left(\int_0^\infty \left(\frac{\omega_1(r,f)_p}{r^s}\right)^q \frac{dr}{r}\right)^{1/q} = \infty.$$

where $\omega_1(r,f)_p := \sup_{|h|\leq r} \left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, \mathrm{d}x\right)^{1/p}$.

So one has to replace $\omega_1(r,f)_p$ with $\omega_2(r,p)_p$ in the case $s \geq 1$.

Hint: Use (12.13) in Lemma 12.9.

Please hand in your solutions for this sheet by Thursday 02/06/2016 into your assistant's pigeonhole. Your teaching assistants will put your corrected sheets in their pigeonholes in F28 in due time.

Dietmar and the team wish you pleasant holidays, so you can return with a fresh mind to study for the exams.