4.1. Removal of singularities [1] Assume $u : B_1(\xi) \setminus \{\xi\} \subset \mathbb{R}^n \to \mathbb{R}$ be a harmonic function which fulfils the condition

$$\lim_{r\to 0} r^{n-1} \sup_{|\xi - x| = r} (|u(x)| + |\nabla u(x)|) = 0.$$

Then u can be extended to a harmonic function on $B_1(\xi)$.

Hint: Prove that u is a weak solution for the Laplace equation on $B_1(\xi)$ by cutting out a small ball $B_r(\xi)$. Use Weyl's Lemma.

Solution: Let for $0 < r < 1$, denote $\Omega_r := B_1(\xi) \setminus B_r(\xi)$. Then we have for all $\varphi \in C_0^\infty(B_1(0))$, that

$$\begin{aligned}\left|\int_{\Omega_r} u\Delta\varphi \, \mathrm{d}x\right| &= \left|\int_{\Omega_r} (u\Delta\varphi - \varphi\Delta u) \, \mathrm{d}x\right| \\ &= \left|\int_{\partial B_r(\xi)} (\varphi\frac{\partial u}{\partial n} - u\frac{\partial \varphi}{\partial n}) \, \mathrm{d}S\right| \\ &\leq \omega_n r^{n-1} \|\varphi\|_{C^1} \sup_{|\xi - x| = r} (|u(x)| + |\nabla u(x)|) \overset{r\to 0^+}{\longrightarrow} 0\end{aligned}$$

Hence, for all $\varphi \in C_0^\infty(B_1(0))$

$$\int_\Omega u\Delta\varphi \, \mathrm{d}x = \lim_{r\to 0^+} \int_{\Omega_r} u\Delta\varphi \, \mathrm{d}x = 0.$$

This proves that $u \in L^1_{loc}(B_1(\xi))$ solves $\Delta u = 0$ weakly, so by Weyl's Lemma u is smooth and harmonic on $B_1(\xi)$.

4.2. Reflection principle [2] Denote by $B_1(0)^+ := \{x \in B_1(0) \subset \mathbb{R}^n : x_n > 0\}$. Assume that $u : B_1(0)^+ \to \mathbb{R}$ is harmonic and admits a continuous extension to $\overline{B_1(0)^+}$ with $u \equiv 0$ on $x_n = 0$. Define an extension $\tilde{u}$ of u to $B_1(0)$ by defining

$$u(x) = -u(\tilde{x}, -x_n)$$

for $x_n < 0$ and where we write $x = (\tilde{x}, x_n) \in \mathbb{R}^{n-1} \oplus \mathbb{R}$. Prove that $\tilde{u} : B_1(0) \to \mathbb{R}$ is harmonic.

Hint: Prove that $\tilde{u}$ is a weak solution for the Laplace equation on $B_1(0)$ by splitting φ into even and uneven parts with respect to x_n and cutting out a symmetric strip around $x_n = 0$. Use Lemma 2 to obtain bounds on ∇u and Weyl's Lemma.

[1] Thank you Christian Beck for giving me the idea for this exercise.
[2] This exercise is a bit longer, but has the same general idea as 4.1.

Solution: We note first that $\tilde{u}$ is continuous, so in particular it is $L^1_{loc}(B_1(0))$. It is also C^2 on $B_1(0)^- := \{x \in B_1(0) \subset \mathbb{R}^n : x_n < 0\}$. For $x_n < 0$, we calculate

$$\partial_i\partial_i\tilde{u}(x) = -\partial_i\partial_i u(\tilde{x}, -x_n)$$
$$\partial_n\partial_n\tilde{u}(x) = -\partial_n\partial_n u(\tilde{x}, -x_n)$$

where $i = 1, \dots, n-1$. So hence $\tilde{u}$ is also harmonic on $B_1(0)^-$.

For $\varphi \in C_0^\infty(B_1(0))$, split it into $\varphi_u(x) := \frac{1}{2}(\varphi(x) - \varphi(\tilde{x}, -x_n))$ and $\varphi_o(x) := \frac{1}{2}(\varphi(x) + \varphi(\tilde{x}, -x_n))$. Then $\varphi_u, \varphi_o \in C_0^\infty(B_1(0))$ and $\varphi = \varphi_u + \varphi_o$. Also φ_o (resp. φ_e) is odd (resp. even) with respect to x_n, so in particular $\varphi_o(\tilde{x}, 0) = 0$.

Denote for $0 < r < 1$ by $\Omega_r := B_1 0 \setminus \{x \in B_1(0) : |x_n| \le r\}$. Denote by S_r^+ and S_r^- the upper and lower component of the boundary of Ω_r. Then we have

$$\begin{aligned}
\int_{\Omega_r} \tilde{u}\Delta\varphi \,\mathrm{d}x &= \int_{\Omega_r} (\tilde{u}\Delta\varphi + \Delta\tilde{u}\varphi) \,\mathrm{d}x \\
&= \int_{S_r^+ \cup S_r^-} (\varphi\frac{\partial\tilde{u}}{\partial n} - \tilde{u}\frac{\partial\varphi}{\partial n}) \,\mathrm{d}S \\
&= \int_{S_r^+ \cup S_r^-} \varphi_e\frac{\partial\tilde{u}}{\partial n} \,\mathrm{d}S + \int_{S_r^+ \cup S_r^-} \varphi_o\frac{\partial\tilde{u}}{\partial n} \,\mathrm{d}S \\
&\quad - \int_{S_r^+ \cup S_r^-} \tilde{u}\frac{\partial\varphi}{\partial n} \,\mathrm{d}S \\
&= I_1 + I_2 + I_3
\end{aligned}$$

We first note that $I_1 = 0$. Indeed, we have that $\partial_n\tilde{u}$ is an even function with respect to x_n, and so $\varphi_e\partial_n\tilde{u}$ is an even function with respect to x_n. However the outward pointing vector for S_r^+ is ∂_n whereas the one for S_r^- is $-\partial_n$. Hence

$$I_1 = \int_{S_r^+ \cup S_r^-} \varphi_e\frac{\partial\tilde{u}}{\partial n} \,\mathrm{d}S = \int_{S_r^+} \varphi_e\frac{\partial\tilde{u}}{\partial n} \,\mathrm{d}S - \int_{S_r^+} \varphi_e\frac{\partial\tilde{u}}{\partial n} \,\mathrm{d}S = 0$$

Along the same line of arguments, we find that $I_2 = 2\int_{S_r^+} \varphi_o\frac{\partial u}{\partial n} \,\mathrm{d}S$. Now u is harmonic on $B_1(0)^+$ and $\mathrm{dist}(\partial B_1(0)^+, S_r^+) = r$. Therefore, by Lemma 2, we get the estimate for $x \in S_r^+$

$$\left|\frac{\partial u}{\partial n}(x)\right| = |\partial_n u(x)| \le \frac{n}{r} \sup_{|x|\le 1,\ x_n \le 2r} |u|$$

On the other hand, as $\varphi_o \in C_0^\infty(B_1(0))$, it is Lipschitz, in particular there is $C_1 > 0$, such that for $x \in S_r^+$

$$|\varphi_o(x) - \varphi_o(\tilde{x}, 0)| \le C_1 r.$$

Combining these two facts, we find that

$$|I_2| \leq 2\,\mathrm{Area}(S_r^+)C_1 r \frac{n}{r} \sup_{|x|\leq 1,\ x_n \leq 2r} |u|\,.$$

Now $\mathrm{Area}(S_r^+) \leq \mathrm{Area}(B_1(0) \cap \{x_n = 0\}) = C_2$. Furthermore, as $u \equiv 0$ on $B_1(0) \cap \{x_n = 0\}$ and $\{|x| \leq 1,\ x_n \leq 2r\}$ is bounded, for every $\epsilon > 0$, there is $r_\epsilon > 0$, such that for all $0 < r < r_\epsilon$, we have

$$\sup_{|x|\leq 1,\ x_n \leq 2r} |u| < \frac{\epsilon}{2nC_1C_2}.$$

This means that for all $0 < r < r_\epsilon$, we have $|I_2| < \epsilon$. I.e. $\lim_{r\to 0^+} I_2 = 0$. Finally, for I_3, we use that $\sup_{S_r^+\cup S_r^-} \left|\frac{\partial\varphi}{\partial n}\right| \leq C_3$ and that $\sup_{S_r^+\cup S_r^-} |\tilde{u}| \leq \frac{\epsilon}{2nC_1C_2}$ whenever $0 < r < r_\epsilon$, so we have $|I_3| \leq 2C_2C_3\frac{\epsilon}{2nC_1C_2} \leq C_4\epsilon$. So we also get $\lim_{r\to 0^+} |I_3| = 0$.

All in all, we have

$$\int_{B_1(0)} \tilde{u}\Delta\varphi\ \mathrm{d}x = \lim_{r\to 0^+} \int_{\Omega_r} \tilde{u}\Delta\varphi\ \mathrm{d}x = \lim_{r\to 0^+} (I_1 + I_2 + I_3) = 0.$$

This proves that $\tilde{u} \in L^1_{loc}(B_1(0))$ solves $\Delta\tilde{u} = 0$ weakly, so by Weyl's Lemma $\tilde{u}$ is smooth and harmonic on $B_1(0)$.

Alternative Solution: [3] By construction, you can verify that $\tilde{u} \in C^1(B_1(0))$ and $\tilde{u} \in C^2(B_1(0) \setminus \{x \in B_1(0) : x_n \neq 0\})$. So we can use integration by part to establish that $\tilde{u}$ is a weak solution to Laplace's equation.

$$\begin{aligned}
\int_{B_1(0)} \tilde{u}\Delta\varphi\ \mathrm{d}x &= -\int_{B_1(0)} \nabla\tilde{u}\cdot\nabla\varphi\ \mathrm{d}x \\
&= -\int_{B_1^+(0)} \nabla\tilde{u}\cdot\nabla\varphi\ \mathrm{d}x - \int_{B_1^-(0)} \nabla\tilde{u}\cdot\nabla\varphi\ \mathrm{d}x \\
&= \int_{B_1(0)^+} \Delta\tilde{u}\varphi\ \mathrm{d}x + \int_{S_0^+} \partial_n\tilde{u}\varphi\ \mathrm{d}\sigma \int_{B_1(0)^-} \Delta\tilde{u}\varphi\ \mathrm{d}x - \int_{S_0^-} \partial_n\tilde{u}\varphi\ \mathrm{d}\sigma \\
&=0.
\end{aligned}$$

where we use in the penultimate line, that the normal vector of S_0^+ is $-\partial_n$ and that vector for S_0^- is ∂_n. In the last line, we used that $\tilde{u}$ is harmonic on $B_1(0)^+ \cup B_1(0)^-$.

4.3. Injectivity of functions Prove that the inclusion $L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ is injective. In other words, prove that any $u \in L^1_{loc}(\Omega)$ with the property $\int_\Omega \varphi u\ \mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(\Omega)$ must be zero almost everywhere.

[3] Suggested to me by Francesco Palmurella.

Hint: This is an exercise in measure theory, and therefore we give extensive hints and references to the script on measure theory by Prof. Salamon. [4] Step 1: Convince yourself, that there is a representative of u which is Borel measurable. Step 2: Prove that $\int_K u = 0$ for all compact sets $K \subset \Omega$ by using cut-off functions and Lebesgue dominated convergence. Step 3: Define two measures $\mu^+(A) = \int_A u^+ \, \mathrm{d}x$ and $\mu^-(A) = \int_A u^- \, \mathrm{d}x$ where $u^+ = \max(u, 0)$ and $u^- = -\min(f, 0)$. Check that these are Borel measures by using Theorem 1.40. Step 4: Use 3.18, to get μ^+ and μ^- are inner regular. Step 5: Prove that μ^+ and μ^- are zero, by decomposing Borel sets A into A^+ and A^- where $u \geq 0$ and $u \leq 0$ and using Step 4 together with Step 2. Step 6: Use Lemma 1.49 to conclude.

Solution: **Step 1:** Let f be a representative of the $u \in L^1_{loc}(\Omega)$. Then we can approximate from below both f^+ and f^- by Lebesgue measurable step functions φ_k. Now, we can modify each step function φ_k on a set of measure zero to get a Borel measurable step function $\tilde{\varphi}_k$ such that $\tilde{\varphi}_k \leq \tilde{\varphi}_{k+1}$, due to the fact that for any Lebesgue measurable set there is some Borel set such that their symmetric difference is a set of measure. Hence $\tilde{f} := \lim_{k\to\infty} \tilde{\varphi}_k$ is Borel measurable and differs from f on a set of measure zero, so hence it also represents u.

Step 2: Fix a compact set K and take a sequence U_k of open sets for $k \in \mathbb{N}$ such that $K \subset U_{k+1} \subset U_k \subset \Omega$ and such that $\operatorname{dist}(\partial U_k, K) \leq \frac{1}{k}$. Then take a cut off function $\varphi_k \in C_0^\infty(\Omega)$ such that $\varphi_k \equiv 1$ on K, $\operatorname{supp}\varphi_k \subset U_k$ and $\operatorname{im}\varphi_k \subset [0, 1]$. As U_1 is bounded, $f\mathbb{1}_{\overline{U_1}}$ is integrable. Thus $|f\varphi_k| \leq f\mathbb{1}_{\overline{U_1}}$ and so by Lebesgue Dominated convergence theorem, we have

$$\int_K u \, \mathrm{d}x = \int_\Omega u\mathbb{1}_K \, \mathrm{d}x = \lim_{k\to\infty} \int_\Omega u\varphi_k \, \mathrm{d}x = 0$$

where the last equality follows by assumption.

Step 3: By theorem 1.40, $\mu^+(A) := \int_A u^+ \, \mathrm{d}x$ for A Borel is a Borel measure, because u^+ is a positive, Borel measurable function. The same goes for μ^-.

Step 4: By theorem 3.18, μ^+ and μ^- are inner regular, because every open set in $\mathbb{R}^n$ is σ compact.

Step 5: Define for a Borel measurable set A, the sets $A^+ := \{x \in A : u(x) \geq 0\}$ and $A^- := \{x \in A : u(x) < 0\}$ are both Borel measurable. Then $A = A^+ \amalg A^-$, and we have

$$\mu^+(A) = \mu^+(A^+) = \sup_{K\subset A^+, K \text{ compact}} \mu^+(K) = \sup_{K\subset A^+, K \text{ compact}} \int_K u \, \mathrm{d}x = 0$$

[4] This can be found at `https://people.math.ethz.ch/~salamon/PREPRINTS/measure.pdf`.

where the first equality uses $u^+ \equiv 0$ on A^-, the second equality uses inner regularity, the third equality uses $u = u^+$ on A^+ and the last one uses Step 2. The same kind of arguments yield $\mu^-(A) = 0$. Hence both Borel measures are zero.

Step 6: We have by Step 5

$$\int_\Omega |u| \, \mathrm{d}x = \mu^+(\Omega) + \mu^-(\Omega) = 0.$$

So by Lemma 1.49, we have that $u = 0$ almost everywhere.

Alternate solution: [5] Fix $K \subset \Omega$ compact and define for $x \in \mathbb{R}^n$

$$\alpha_K(x) = \begin{cases} \dfrac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, x \in K \\ 0 & \text{else} \end{cases}$$

Notice that $|\alpha_K(x)| \leq 1$ for all $x \in \mathbb{R}^n$. You can now apply a mollifier to it, i.e.

$$\alpha_k^\epsilon := \alpha_k * \rho_\epsilon \in C^\infty(\mathbb{R}^n)$$

and for $\epsilon > 0$ sufficiently small, we have $\operatorname{supp} \alpha_K^\epsilon \subset \Omega$. Furthermore, we have that $\alpha_K^\epsilon \to \alpha_K$ almost everywhere. Furthermore, we have

$$|\alpha_K^\epsilon(x) u(x)| \leq |\alpha_K^\epsilon(x)| \, |u(x)| \leq |u(x)| \,,$$

because $|\alpha_K^\epsilon(x)| \leq \int_{\mathbb{R}^n} \rho_\epsilon(y) \, |\alpha_K(x-y)| \, \mathrm{d}y \leq \int_{\mathbb{R}^n} \rho_\epsilon(y) \, \mathrm{d}y = 1$. Therefore, we may apply the dominated convergence theorem to $\alpha_K^\epsilon(x)u(x)$ which converges point-wise almost everywhere to $|u(x)| \, \mathbb{1}_K$. Therefore, by assumption, we have

$$0 = \lim_{\epsilon \to 0} \int_\Omega \alpha_K^\epsilon u \, \mathrm{d}x = \int_\Omega \lim_{\epsilon \to 0} \alpha_K^\epsilon \, \mathrm{d}x = \int_K |u| \, \mathrm{d}x.$$

Thus, we have $u = 0$ almost everywhere in K. As K was arbitrary and Ω is σ compact, we conclude that $u = 0$ almost everywhere on Ω.

4.4. Equivalent norms Let $u \in W^{k,p}(\mathbb{R}^n)$, which means that u has weak derivatives $\partial^\alpha u \in L^p(\mathbb{R}^n)$ for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq k$. Prove that the norms

$$\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \qquad \text{and} \qquad \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

[5] Suggested to me by Francesco Palmurella.

are equivalent.

Solution: We define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^N$, where $N = \#\{\alpha : |\alpha| \leq k\}$. Namely,

$$\|x\|_1 = \sum_{|\alpha|\leq k} |x_\alpha| \qquad \text{and} \qquad \|x\|_2 = \left(\sum_{|\alpha|\leq k} |x_\alpha|^p\right)^{\frac{1}{p}}$$

for all $x \in \mathbb{R}^N$. As all the norms on $\mathbb{R}^N$ are equivalent, we have that there is a constant $C \geq 1$ such that

$$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$$

for all $x \in \mathbb{R}^N$. Hence, we get for $x = (\|\partial^\alpha u\|_{L^p(\mathbb{R}^n)})_{|\alpha|\leq k}$, the wanted equivalence of the norms.

4.5. Reflexivity of Sobolev spaces Prove that $W^{k,p}(\mathbb{R}^n)$ is reflexive for all $k \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$.

Hint: Recall from FA I, that the spaces $L^p(\mathbb{R}^n, \mathbb{R}^N)$ are reflexive for $1 < p < \infty$.

Solution: The spaces $W^{k,p}(\mathbb{R}^n)$ can be injected into $L^p(\mathbb{R}^n, \mathbb{R}^N)$ for $N = \#\{\alpha : |\alpha| \leq k\}$. Call this map ι. This space can be equipped with several equivalent norm, and we choose to equip it with the norm $\|u\|_{L^p(\mathbb{R}^n,\mathbb{R}^N)} = \sum_{i=1}^N \|u_i\|_{L^p(\mathbb{R}^n)}$. This makes the inclusion ι isometric. Now $W^{k,p}(\mathbb{R}^n)$ is complete, hence the image of the inclusion ι is a complete subspace of a complete space, so $\operatorname{im}\iota$ is closed. Now we know from FA I, that any closed subspace of a reflexive space is again reflexive. Thus we deduce from the reflexivity of $L^p(\mathbb{R}^n, \mathbb{R}^N)$ the reflexivity of $\operatorname{im}\iota$. Hence, $W^{k,p}(\mathbb{R}^n)$ is reflexive as ι is an isometric embedding.