6.1. Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$.

Prove that the following are equivalent:

(i) $u \in W^{1,\infty}_{loc}(\Omega)$.

(ii) u is locally Lipschitz.

Hint: For (i) implies (ii) use mollifiers $\rho_\delta(x) := \frac{1}{\delta^n}\rho(x/\delta)$ with $\operatorname{supp}\rho \subset B_1(0)$ and estimate the Lipschitz constant for $u_\delta = u * \rho_\delta$ as $\delta \to 0$. For (ii) implies (i), consider a fixed vector $\xi \in \mathbb{R}^n$ and define the difference quotient

$$u_j(x) := j\left[u(x + \frac{\xi}{j}) - u(x)\right].$$

Prove that there is $u^\xi \in L^\infty_{loc}$, such that a subsequence of u_j weakly converges to u^ξ in L^2_{loc}. Show that

$$\int_\Omega u^\xi \varphi = -\int_\Omega u \partial_\xi \varphi$$

for any $\varphi \in C^\infty_0(\Omega)$ by proving a similar equality for u_j and taking the limit.

6.2. Let $\Omega \subset \mathbb{R}^2$ be defined by

$$\Omega := \Omega_0 \cup \bigcup_{m=0}^{\infty} \Omega_m$$

$$\Omega_0 := \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < \frac{1}{2}\}$$

$$\Omega_m := \{(x, y) \in \mathbb{R}^2 : \frac{1}{2^{2m+1}} < x < \frac{1}{2^{2m}}, \frac{1}{2} \leq y < 1\}$$

(a) Show that the embedding $W^{1,2}(\Omega) \to L^2(\Omega)$ is not compact.

(b) Show that $W^{1,2}(\Omega)$ is not a subset of $L^q(\Omega)$ for $q > 2$.

6.3. Show that $C^\infty(\overline{\Omega})$ is not dense in $W^{1,p}(\Omega)$ for $p \geq 1$ where:

(a) $\Omega = (-1, 0) \cup (0, 1)$.

(b) $\Omega := \{(x, y) \in \mathbb{R}^2 : |(x, y)| < 1\} \setminus \{(x, 0) \in \mathbb{R}^2 : 0 \leq x < 1\}$.

Hint: For (b), prove that for $0 < \epsilon \leq 1$ and for any smooth function $\varphi : [-\epsilon, \epsilon] \to \mathbb{R}$, one has

$$\int_{-\epsilon}^{0} |\varphi(t)| \, \mathrm{d}t + \int_{0}^{\epsilon} |1 - \varphi(t)| \, \mathrm{d}t + \int_{-\epsilon}^{\epsilon} |\varphi'(t)| \, \mathrm{d}t \geq \epsilon.$$

Then consider a function $u \in W^{1,p}(\Omega)$ which cannot be extended to a continuous function on $B_1(0)$ and find a contradiction once you try to approximate it by smooth functions.

6.4. Let $\Omega \subset \mathbb{R}^n$ be open. Let $u_n \in W^{k,p}(\Omega)$ be a Cauchy sequence with $u_n \to u$ in $L^p(\Omega)$. Prove that $u \in W^{k,p}(\Omega)$ and that $u_n \to u$ in $W^{k,p}(\Omega)$.

6.5. Let $\Omega = \mathbb{R}^n$, $p \geq 2$ and $u : \mathbb{R}^n \to \mathbb{R}$ in $W^{2,p}(\Omega)$.

(a) For $n = 1$, prove that

$$\int_{\mathbb{R}} |u'|^p \leq C(p) \int_{\mathbb{R}} |uu''|^{\frac{p}{2}}$$

(b) Prove that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p) \, \|u\|_{L^p(\Omega)}^{1/2} \, \|u\|_{W^{2,p}(\Omega)}^{1/2}$$

(c) Prove that

$$\|u\|_{W^{1,p}(\Omega)} \leq C(n,p) \, \|u\|_{L^\infty(\Omega)}^{1/2} \, \|u\|_{W^{2,\frac{p}{2}}(\Omega)}^{1/2}$$

Hint: For (a) start with a compactly supported smooth function u and consider $v := u' \, |u'|^{p-2}$. Then use the integration by part formula for $w = uv$ and the generalised Hölder inequality.

This sheet will be discussed after the Easter break. Please hand in your solutions for this sheet by Monday 11/04/2016.