8.1. Composition of Sobolev functions. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^1 boundary and $1 \leq p < \infty$. Pick $f \in C^1(\mathbb{R})$ with $f' \in L^\infty$. Prove that for $u \in W^{1,p}(\Omega)$, we also have $f \circ u \in W^{1,p}(\Omega)$ and

$$\partial_i(f \circ u) = f'(u)\partial_i u.$$

Hint: Approximate u by smooth functions.

8.2. The absolute value of a Sobolev function. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^1 boundary and $1 \leq p < \infty$. Prove that for $u \in W^{1,p}(\Omega)$, that we also have $|u| \in W^{1,p}(\Omega)$ and that

$$\partial_i |u| = \operatorname{sgn}(u)\partial_i u.$$

Hint: Use Exercise 8.1 with $f_\epsilon(x) := \sqrt{x^2 + \epsilon^2} - \epsilon$.

8.3. Iterated Calderòn–Zygmund. Prove that for all $m, n \in \mathbb{N}$ and $1 < p < \infty$, there is a constant $C > 0$ such that

$$\left\|\partial^{2m} u\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\Delta^m u\right\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

8.4. Schwartz space and Fourier transform. [1] The goal of this exercise is to define the Schwartz space and study the Fourier transform on it. We define norms for $k \in n$ on $C^\infty(\mathbb{R}^n)$ by

$$\|u\|_k := \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha u(x)| \, (1 + |x|)^k.$$

and define

$$\mathcal{S}(\mathbb{R}^n) := \{u \in C^\infty(\mathbb{R}^n) : \|u\|_k < \infty \text{ for all } k \in \mathbb{N}\}.$$

This is a complete, topological vector space with respect to the distance function

$$d(u, v) := \sum_{k \geq 1} 2^{-k} \frac{\|u - v\|_k}{1 + \|u - v\|_k}$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$.[2]

[1] This and the next Exercise are simply a lot of checking, don't worry ;)
[2] This is an example of a Fréchet space, a way of generalising Banach spaces.

(a) Prove that $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \bigcap_{1\le p\le\infty} L^p(\mathbb{R}^n)$. We state as a fact that these inclusions are continuous and have dense image in the respective distance functions.

Hint: Use the fact that $\frac{1}{(1+|x|)^{mp}}$ is in $L^1(\mathbb{R}^n)$ whenever $mp > n$.

(b) Prove that for $u, v \in \mathcal{S}(\mathbb{R}^n)$ and for P a polynomial, we get that Pu, uv and $\partial^\alpha u$ are elements of $\mathcal{S}(\mathbb{R}^n)$.

(c) Prove for $u \in \mathcal{S}(\mathbb{R}^n)$ that its Fourier transform $\hat{u} \in \mathcal{S}(\mathbb{R}^n)$. Also prove that if $\lim_{k\to\infty} d(u_k, u) = 0$ for $u_k, u \in \mathcal{S}(\mathbb{R}^n)$, then also $\lim_{k\to\infty} d(\hat{u_k}, \hat{u}) = 0$.

(d) Prove that the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism of topological vector spaces.

Hint: Use the Fourier inverse formula, which says that $u = \tilde{\mathcal{F}}(\mathcal{F}(u))$ for $u \in \mathcal{S}(\mathbb{R}^n)$ and $\tilde{\mathcal{F}}(u)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\hat{u}(\xi)\ \mathrm{d}\xi$.

8.5. A generalised Fourier transform.

(a) Prove that for $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} \hat{u}v = \int_{\mathbb{R}^n} u\hat{v}.$$

By the same token prove

$$\int_{\mathbb{R}^n} u\bar{v} = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}.$$

This gives you Plancherel's identity

$$\|u\|_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2}\|\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

(b) Extend $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ in a unique way to an isomorphism $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with

$$(2\pi)^{-n/2}\|\mathcal{F}(u)\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Prove that this agrees with $\hat{u}$ for $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Hint: For the second statement, start with u having compact support and mollifiers to get $\hat{u} = \mathcal{F}(u)$ on every compact set, next try to deduce the general case from this special case.

(c) Introduce $\mathcal{S}(\mathbb{R}^n)'$ the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$. As $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, this is a subset of distributions called tempered distributions. They should be thought of as the distributions you can apply Fourier transform to. Prove that

$$T_f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto \int_{\mathbb{R}^n} fu$$

is a tempered distribution for all $f \in L^p(\mathbb{R}^n)$ and $1 \leq p \leq \infty$.

(d) Introduce for $T \in \mathcal{S}(\mathbb{R}^n)'$, its Fourier transform $\mathcal{F}(T)$ by setting

$$\langle \mathcal{F}(T), u \rangle := \langle T, \hat{u} \rangle .$$

Prove that $\mathcal{F}(T) \in \mathcal{S}(\mathbb{R}^n)'$.

(e) Prove that for $f \in L^2(\mathbb{R}^n)$, we have

$$\mathcal{F}(T_f) = T_{\mathcal{F}(f)}.$$

and that for $f \in L^1(\mathbb{R}^n)$, we have

$$\mathcal{F}(T_f) = T_{\hat{f}}.$$

So the Fourier transform on tempered distributions generalises both notions of Fourier transform.

(f) Prove that for $T \in \mathcal{S}(\mathbb{R}^n)'$, the functions

$$x^\alpha T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto T(x^\alpha u)$$

and

$$\partial^\beta T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : u \mapsto (-1)^{|\beta|} T(\partial^\beta u)$$

are tempered distributions for all multi-indices α, β. Prove that $\mathcal{F}(\partial_j T) = ix_j \mathcal{F}(T)$ and $\mathcal{F}(ix_j T) = -\partial_j \mathcal{F} T$.

8.6. Calderòn–Zygmund inequality via multipliers. We reprove the Calderòn–Zygmund inequality by using the Mikhlin multiplier theorem. Prove the following steps. Let $f \in C_0^\infty(\mathbb{R}^n)$.

(a) Prove that the function $K_j * f$ defines a tempered distribution in the usual way. Call this distribution T_j.

(b) Prove that $-\sum_{i=1}^n (x_i^2 \mathcal{F}(T_j)) = i\xi_i T_{\hat{f}}$, by using the identity $\Delta(K_j * f) = \partial_j f$.

(c) Prove that $\mathcal{F}(\partial_i(K_j * f)) = m_{ij}\hat{f}$ where $m_{ij}(\xi) := \frac{\xi_i \xi_j}{|\xi|^2}$.

(d) Deduce the Calderòn–Zygmund inequality from Mikhlin multiplier theorem.

Please hand in your solutions for this sheet by Monday 25/04/2016.