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Numerische Mathematik I

ETH Zürich
D-MATH

Homework Problem Sheet 11

Problem 11.1 Identification of Zeros

Consider

$$f(x) = \exp(2x) - \sin(x) - 2.$$

(11.1a) One example of a fixed point equation to determine the zeros of f is $f(x) + x = x$. Find two further fixed-point equations to determine the zeros of $f(x)$.

Solution:

(11.1b) Do your proposed maps Φ_1, Φ_2 satisfy the conditions of the Banach fixed-point theorem? Show that $f(x)$ has a unique real zero.

Solution: We verify the conditions of Banach's fixed-point theorem on a yet to be determinted interval I.

We are able to show that $\Phi_1'(x) > 1\ \forall x \in \mathbb{R}$, by transforming the equation $\Phi_1'(x) = 1$ to a quadratic equation

2. We know $\max(\Phi_2(x)) = \Phi_2(\pi/2 + 2\pi\mathbb{Z}) = \frac{1}{2}\ln(3)$ and $\min(\Phi_2(x)) = \Phi_2(3\pi/2 + 2\pi\mathbb{Z}) = 0$. Therefore Φ_2 is a function that maps $I = [0, \frac{1}{2}\ln(3)]$ to itself.

Hence all the conditions of Banach's fixed-point theorem are fulfilled on the interval $I = [0, \frac{1}{2}\ln(3)]$. Thus f has one unique root in this interval.
Outside of this interval f does not have any zeros since

(11.1c) Give a number $n \in \mathbb{N}$, s.t. with the starting point $x^{(0)} := 0.1$, the absolute error of the nth iterate in determining the root of $f(x)$ (on the fixed-point equation found in subproblem (11.1b)) is less than 10^{-8}.

Solution: From $x^{(0)} = 0.1$ follows $x^{(1)} = \frac{1}{2}\ln(\sin(0.1) + 2) \approx 0.3709$. The a priori error estimate in [NMI, Thm. 5.10] then states that

Problem 11.2 MATLAB: Fixed-point, Secant and Newton's Method

A sphere with radius $R = 1$ and density $\rho \in (0, 1)$ is swimming in water (density of water $\rho_W = 1$). We want to figure out how deep the sphere immerses $h = h(\rho)$.

(11.2a) Show that if all the forces compensate then the cubic equation $h^3 - 3h^2 + 4\rho = 0$ holds.

HINT: The weight force of the sphere must equal the buoyant force caused by the displaced water:

$$m_K g = \rho_W V_{KS} g,$$

where V_{KS} denotes the volume of the immersed part of the sphere.

(11.2b) Solve the equation by using the fixed-point iteration $h_{k+1} = \sqrt{(h_k^3 + 4\rho)/3}$, the secant method and Newton's method for $\rho = 0.0001, 0.05, 0.4, 0.6, 0.95, 0.9999$. To do this implement the methods in MATLAB and use $h_0 = R$ as initial value. Stop the iteration once $|h_k - h_{k-1}| \leq 10^{-5}$ holds or at the latest when $k = 10^4$.

(11.2c) Explain the differences in the way the iterations converge depending on ρ.

Listing 11.1: Output for Testcalls for **??**

```
Newton

   Density  Immersion Depth Iterations
   0.0001    0.0115693352     10.00
   0.0500    0.2707007243      5.00
   0.4000    0.8658621543      3.00
   0.6000    1.1341378457      3.00
   0.9500    1.7292992757      5.00
   0.9999    1.9884306648      9.00

Secant

   Density  Immersion Depth Iterations
   0.0001    0.0115693352     13.00
   0.0500    0.2707007243      7.00
   0.4000    0.8658621544      3.00
   0.6000    1.1341378455      3.00
   0.9500    1.7292992719      6.00
   0.9999    1.9884306528     12.00

Fixed-point

   Density  Immersion Depth Iterations
   0.0001    0.0115693352      8.00
   0.0500    0.2707008788      9.00
   0.4000    0.8658652884     13.00
   0.6000    1.1341251516     16.00
   0.9500    1.7291918167     53.00
   0.9999    1.9854194221    362.00
```

Problem 11.3 Modified Newton Method

(11.3a) Determine the order and condition of the root $x^\star = 0$ of $f(x) = x^4$.

Solution: The multiplicity of the root is 4 since $f^{(4)}$ is the very first derivative that isn't equal to zero when evaluated at $x = 0$. The condition of the root is given by the smallest κ that satisfies $|\widetilde{x}^\star - x^\star| \leq \kappa|\widetilde{f}(x) - f(x)|$, where $x^\star$ denotes the root of f and $\widetilde{x}^\star$ denotes the root of the perturbed function $\widetilde{f}$. Let's assume that the perturbation is bounded by

$$|\widetilde{f}(x) - f(x)| \leq \epsilon.$$

Therefore we have at $x = \widetilde{x}^\star$

We use the Taylor expansion of $f(\widetilde{x}^\star)$ and the fact that all derivatives up to the m-the derivative are equal to zero (in this case $m = 4$):

This way we do not get a constant κ, we see however that the distance between the roots becomes bigger as m becomes greater since $\epsilon^{1/m}$ is much greater than ϵ for a small ϵ. Even though the second factor does reduce this effect, the condition of the root can get much worse. In this case we get

(11.3b) Determine Newton's method $x \mapsto \Phi(x)$ in order to find the root $x^\star$. Show that Φ converges for all $x \in \mathbb{R}$. What is the order of convergence? Is the convergence quadratic? Explain your answer.

Solution:

(11.3c) Let $f \in \mathcal{C}^\infty(\mathbb{R})$, $m \in \mathbb{N}$ and $x^\star \in \mathbb{R}$ with

$$f(x^\star) = f'(x^\star) = \cdots = f^{(m-1)}(x^\star) = 0 \neq f^{(m)}(x^\star).$$

Show that:

- We have $\Phi'(x^\star) = 1 - 1/m$ for Newton's method Φ considering the function f.
- The method defined by

$$x_{n+1} := \Psi(x_n), \quad n = 0, 1, \ldots \quad \text{with} \quad \Psi : \begin{cases} U \to \mathbb{R}, \\ x \mapsto x - m\frac{f(x)}{f'(x)} \end{cases}$$

converges quadratically in a neighbourhood U of $x^\star$.

Solution: Since $x^\star$ is a root of multiplicity m of f, we can denote f as

$$f(x) = (x - x^\star)^m g(x) \tag{11.3.1}$$

with $g(x^\star) \neq 0$. Therefore we get

- For $\Phi(x) = x - f(x)/f'(x)$ it follows

- For $\Psi(x) = x - m\frac{f(x)}{f'(x)}$ we have

Remark: WLOG $x^\star = 0$. In order to use [NMI, Thm. 5.9] we need continuity of

$$\begin{aligned}
&\Psi''(x)(mg(x) + xg'(x))^3 = \\
&= x^2 g'(x)^2(2mg'(x) + xg''(x)) + mg(x)^2\big(2g'(x) + x\big(4g''(x) + xg^{(3)}(x)\big)\big) + \\
&+ xg(x)\big(-4mg'(x)^2 - 2x^2 g''(x)^2 + xg'(x)\big(-3mg''(x) + xg^{(3)}(x)\big)\big)
\end{aligned}$$

in a sufficiently small neighbourhood of $x^\star$; this holds if $x^k g^{(k)}(x)$, $k = 0, 1, 2, 3$ is bounded in a neighbourhood of $x^\star = 0$. This is the case for instance if f is at least three times continuously differentiable.

Problem 11.4 MATLAB: Newton's Method for the Eigenvalue Problem

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ have only real eigenvalues and eigenvectors. We are looking for a pair of eigenvector and eigenvalue $(\mathbf{v}, \lambda) \in \mathbb{R}^n \times \mathbb{R}$ of $\mathbf{A}$ that satisfies the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ under the constraint that $\mathbf{v}^\top\mathbf{v} = 1$.

(11.4a) Formulate Newton's method for the non-linear system of equations $\mathbf{F}(\mathbf{v}, \lambda) = 0$.

HINT: We can rewrite the problem in the form $\mathbf{F}(\mathbf{v}, \lambda) = \mathbf{0}$, where $\mathbf{F} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$. We define

$$\mathbf{F}(\mathbf{v}, \lambda) = \begin{pmatrix} \mathbf{A}\mathbf{v} - \lambda\mathbf{v} \\ -\mathbf{v}^\top\mathbf{v} + 1 \end{pmatrix}.$$

Therefore we have that the root $(\mathbf{v}, \lambda)$ of $\mathbf{F}$ satisfies $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v}^\top\mathbf{v} = 1$. If we consider that $\mathbf{v}$ is real, we notice that the latter is equivalent to the normalization $\|\mathbf{v}\|_2 = 1$.

(11.4b) Implement a MATLAB function

```
[l, T] = fixpkt(phi, v, l, tol, maxit)
```

that performs a fixed point iteration on the pair $(\mathbf{v}, \lambda)^\top$ and takes as input the Newton iteration function `phi` found in subproblem (11.4a), the initial guess (`v`, `l`), the maximum number of iterations `maxit` and the tolerance `tol`. As stopping criterion, use the distance between two consecutive approximations for the eigenvalue. Intermediate values of the eigenvalues have to be stored in the output vector `T` except for the last one `l`.

Test your code for:

```
m = 5; n = m^2;
A = gallery('poisson',m);
v = 1/2/n*ones(n,1); l = 0;
[l, T] = fixpkt(phi, v, l, 1e-10, 25)
```

and a suitable implementation of `phi`.

Listing 11.2: Test call output for subproblem (11.4b)

```
l = 2.2679

T =       0    30.2953    14.8882     6.7604     2.3736     1.1300
     2.2555     2.2669     2.2676     2.2679     2.2679     2.2679
```

(11.4c) Using the function `fixpkt` from subproblem (11.4b), determine (numerically) the local convergence rate of the Newton iteration from subproblem (11.4a) for the following choice of the matrix $\mathbf{A} = \mathbf{A}_i$, $i \in \{1, 2, 3\}$, where

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Use as initial guess $\mathbf{v}^{(0)} = (-1, 1)^\top$, $\lambda^{(0)} = 0$ and display in a semilog plot the deviation $|\lambda_k - \lambda_K|$, $0 \le k < K$. In this case $K \ge 0$ is minimal, in such a way that either $|\lambda_{K-1} - \lambda_K| \le \texttt{tol} = 10^{-14}$ or $K \ge \texttt{maxit} = 25$ holds. What can you infer?

(11.4d) Prove what you have numerically observed in subproblem (11.4c): The Newton iteration from subproblem (11.4a) converges locally quadratically if λ has algebraic multiplicity *one*.

HINT: Look at [NMI, Thm. 5.20]. Consider the geometric and algebraic multiplicity of λ and distinguish among cases.

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

References

[NMI] Lecture Notes for the course "Numerische Mathematik I".

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