D-MATH
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Algebra I

HS 2013

Serie 1

Groups, Cyclic groups

1. Prove the following properties of inverses.
   (a) If an element a has both a left inverse l and a right inverse r, then $r = l$, a is invertible and r is its inverse.
   (b) If a is invertible, its inverse is unique.
   (c) Inverses multiply in the opposite order: if a and b are invertible, then the product ab is invertible and $(ab)^{-1} = b^{-1}a^{-1}$.

2. Let $\mathbb{N}$ denote the set $\{1, 2, 3, \ldots\}$ of natural numbers, and let $s : \mathbb{N} \to \mathbb{N}$ be the *shift* map, defined by $s(n) = n + 1$. Prove that s has no right inverse, but that it has infinitely many left inverses. Deduce that in a set with a law of composition, an element may have a left inverse, even if it is not invertible.

3. Make a multiplication table for the symmetric group S_3.

4. In which of the following cases is H a subgroup of G?
   (a) $G = GL_n(\mathbb{C})$ and $H = GL_n(\mathbb{R})$.
   (b) $G = \mathbb{R}^\times$ and $H = \{1, -1\}$.
   (c) $G = \mathbb{Z}^+$ and H is the set of positive integers.
   (d) $G = \mathbb{R}^\times$ and H is the set of positive reals.
   (e) $G = GL_2(\mathbb{R})$ and H is the set of matrices $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, with $a \neq 0$.

5. In the definition of a subgroup H of a group G the identity element of H is required to be the identity of G. Prove that it would be enough to require that H has an identity element x: in that case x must be the identity of G.

6. Describe all groups G that contain no proper subgroup.

7. Recall from Gerd Fisher, Lineare Algebra, Section 2.7 that an elementary matrix of the first kind in $GL_n(\mathbb{R})$ is a diagonal matrix $S_i(\lambda)$ having all diagonal elements equal to one apart from the i-th that is equal to λ for some real number λ different from 0; and recall that an elementary matrix of the second kind is a matrix $Q_i^j(\lambda)$ such that every diagonal entry of $Q_i^j(\lambda)$ is 1 and such that all other entries are zero apart from the entry in the i-th row and j-th column that is equal to λ, for some real number λ and some pair (i, j) with i different from j and between 1 and n. Prove that:

(a) The elementary matrices of the first and second kind generate $GL_n(\mathbb{R})$.

(b) The elementary matrices of the second kind generate $SL_n(\mathbb{R})$. Do the 2×2 case first.

More challenging problems

The homophonic group

1. By definition, English words have the same pronunciation if their phonetic spellings in the dictionary are the same. The *homophonic* group $\mathcal{H}$ is generated by the letters of the alphabet, subject to the following relations: English words with the same pronunciation represent equal elements of the group. Thus $be = bee$, and since $\mathcal{H}$ is a group, we can conclude that $e = 1$ (why?). Try to determine the group $\mathcal{H}$.