D-MATH Algebra I HS 2013
Prof. Brent Doran

Test 2

Group theory

1. In each of the following cases give an example, or say that no such example exists. Briefly justify your answers.

   (a) A p-group that is not abelian.
   Solution The quaternion group Q is not abelian and has order 8.

   (b) A group whose commutator subgroup is trivial.
   Solution Any abelian group, for example $\mathbb{Z}$.

   (c) An infinite group that is finitely presented.
   Solution Again $\mathbb{Z}$.

   (d) A representation of a group that is not a direct sum of irreducible representations.
   Solution The representation of $\mathbb{Z}$ in $\mathrm{GL}_2(\mathbb{R})$ given by $z \mapsto \left[\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right]$.

   (e) A simple group that is not cyclic.
   Solution A_5.

   (f) A subgroup of a free group on two generators that is isomorphic to a free group on 3 generators.
   Solution The subgroup generated by aba, ab^2a, ab^3a.

   (g) Groups A, B, C such that $A \lhd B \lhd C$, but A is not normal in C.
   Solution $C = S_4$, $B = \{(12)(34), (13)(24), (14)(23), e\}$, $C = \{(12)(34), e\}$.

   (h) A short exact sequence that is not split.
   Solution $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$.

   Bloopers,some answers we got to read :

   (a) D_3, S_3, $\begin{pmatrix} a & b \\ & 1 \end{pmatrix}$ as p-groups

   (b) A_4, S_3, D_5, D_7 have trivial commutator subgroup (aka they are abelian)

   (c) Some finitely presented infinite groups : S_3, $\mathbb{Z}/2\mathbb{Z}$

   (d) A non-cyclic simple group : $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, or $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ 'since it is of order 60'

2. Let G be a group. A subgroup H of G is *characteristic* if $\varphi(H) = H$ for any automorphism φ of G.

(a) Prove that the center $Z(G)$ is a characteristic subgroup of G.
Solution An element g is in the center of G if and only if for every element h in G $hgh^{-1} = g$. Let now g be in the center, and let h be any element of G. Denote by k the unique element of G such that $\varphi(k) = h$, we have $h\varphi(g)h^{-1} = \varphi(k)\varphi(g)\varphi(k)^{-1} = \varphi(kgk^{-1}) = \varphi(g)$. And this concludes the proof.

(b) Prove that any characteristic subgroup of G is in particular a normal subgroup. *Hint*: consider the action of a group on itself by conjugation.
Solution Let g be an element of G, let us consider the automorphism φ_g of G given by $\varphi_g(z) = gzg^{-1}$. Let now N be a characteristic subgroup of G, since $\varphi(N) = N$ for any automorphism of G, we get in particular that $\varphi_g(N) = gNg^{-1} = N$.

(c) Give an example of a normal subgroup of the Quaternion group that is NOT characteristic.
Solution The subgroup of Q generated by i is normal in Q, but is not characteristic: indeed let φ be the automorphism of Q given by $\varphi(i) = j, \varphi(j) = k, \varphi(k) = i$ where i, j, k are elements of Q with $ijk = 1$. Then $\varphi(\langle i \rangle) = \langle j \rangle$.

3. Let a group G act transitively on a set X of at least two elements.

(a) Show that for every pair of points x, y the cardinality of the stabilizers G_x, G_y are equal.
Solution: In fact let g be an element of g such that $gx = y$. Such element exists since the action of G on X is transitive. We claim that $gG_xg^{-1} = G_y$: indeed $gG_xg^{-1}y = gG_xx = gx = y$ an this shows that $gG_xg^{-1} \subseteq G_y$, the other inclusion follows from the fact that $g^{-1}y = x$ hence $g^{-1}G_yg \subseteq G_x$. In particular this implies that $|G_x| = |G_y|$.

(b) Deduce from the counting formula that

$$|G| = \sum_{x \in X} |G_x| = \sum_{g \in G} \text{fix}(g).$$

Solution The counting formula gives $|G| = |G_x||O_x|$ where O_x is the cardinality of the orbit of x. By transitivity of the action this implies that $|G| = |G_x||X|$ moreover part (a) gives $|G_x||X| = \sum_{x \in X} |G_x|$. The second equality follows from the fact that both $\sum_{x \in X} |G_x|$ and $\sum_{g \in G} \text{fix}(g)$ are equal to the number of pairs $(g, x) \in G \times X$ such that x is a fixpoint of G.

(c) Deduce from the previous formula that there exists an element of g acting without fixpoint.
Solution Indeed the identity of G has $|X|$ fixpoint (since it act as the identity). Since $|X| > 1$ we get that at least one element $g \in G$ should have no fixed points so that the equality of (b) can be satisfied.

4. Let $p < q$ be primes. Prove that any nonabelian group of order pq is a semidirect product.
Solution This is an application of the Sylow theorems. We compute easily from the third Sylow theorem that there is only one Sylow q-subgroup of G. In particular, this subgroup must be normal in G and we will denote it by N. Let H be a p-subgroup of G. We show that G decomposes into the semidirect product of H and N. The intersection $H \cap N$ must be trivial, since as a subgroup of both H and N, its order is a common divisor of p and q. Then $H \cdot N$ is a subgroup of G that actually spans G, i.e. $G = H \cdot N$ since

$$|H \cdot N| = \frac{|H||N|}{|H \cap N|} = |G|$$

(cf. Serie 6). Hence the claim follows, and because H and N are both finite groups of prime order, they need to be isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ respectively. That is,

$$G \simeq \mathbb{Z}_p \ltimes \mathbb{Z}_q.$$

5. (a) Let G be a nonabelian group of order 55, determine the class equation of G (you can use without proof that such a group exists).
Solution We first note that s_5, the number of Sylow 5-subgroups, must be 11 and not 1. Otherwise G would be the *direct* product of abelian groups and hence abelian itself.

Recall that by the second Sylow theorem, the 5-subgroups of G are all conjugate. In particular, the conjugacy class of any generator of H has 11 elements. Because H has exactly 4 generators, we already have 4 conjugacy classes of 11 elements.

Next, let x be a generator of N. We want to compute the conjugacy class of x. The conjugate of x by a generator of H cannot be x, since otherwise the centralizer of N would be the whole group G. This implies that the conjugacy class of x consists of 5 elements, hence there are two more conjugacy classes, each consisting of 5 elements.
This gives the class equation

$$55 = 1 + 5 + 5 + 11 + 11 + 11 + 11.$$

(b) The character table of $\mathbb{Z}/5\mathbb{Z}$ is

	$C(0)$	$C(1)$	$C(2)$	$C(3)$	$C(4)$
1	1	1	1	1	1
χ_1	1	ξ	ξ^2	ξ^3	ξ^4
χ_2	1	ξ^2	ξ^4	ξ^3	ξ
χ_3	1	ξ^3	ξ	ξ^4	ξ^2
χ_4	1	ξ^4	ξ^3	ξ^2	ξ

Use this information to fill in 5 rows of the character table of G.

Solution N is normal in G and $\mathbb{Z}/5\mathbb{Z}$ is the quotient G/N. In particular for every irreducible representation χ of G/N we get an irreducible representation $\overline{\chi}$ of G that is obtained precomposing with the quotient map.

	$C(1)$	$C(x)$	$C(x^2)$	$C(y)$	$C(y^2)$	$C(y^3)$	$C(y^4)$
1	1	1	1	1	1	1	1
$\overline{\chi}_1$	1	1	1	ξ	ξ^2	ξ^3	ξ^4
$\overline{\chi}_2$	1	1	1	ξ^2	ξ^4	ξ^3	ξ
$\overline{\chi}_3$	1	1	1	ξ^3	ξ	ξ^4	ξ^2
$\overline{\chi}_4$	1	1	1	ξ^4	ξ^3	ξ^2	ξ
$\overline{\chi}_5$							
$\overline{\chi}_6$							

(c) How many other irreducible representations are there? Determine their dimensions.

Solution We know from the theory that there are as many irreducible representations as conjugacy classes. In particular there are two other irreducible representations. Moreover it was shown in class that, denoting by n_i the dimension of the i-th irreducible representation of G we have

$$|G| = \sum_i n_i^2.$$

Moreover the dimensions of the irreducible representation must divide the order of the group. This implies that $n_5^2 + n_6^2 = 55 - 5 = 5$ hence $n_5 = n_6 = 5$.