D-MATH | Algebra I | HS 2013
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Solution 5

Dihedral groups, permutation groups, discrete subgroups of M_2, group actions

1. Write an explicit embedding of the dihedral group D_n into the symmetric group S_n.
   Solution Let us consider a regular n-gon Δ in the plane $\mathbb{R}^2$ whose center is the origin and having the point $(1,0)$ as a vertex. The subgroup Γ of M_2 consisting of the isometries of the plane that preserve Δ is a dihedral group D_n, generated by the rotation ρ_θ of angle $\theta = \frac{2\pi}{n}$ and the reflection r along the horizontal axis. Let us denote by $1,\dots,n$ the vertices of Δ, numbered in anticlockwise order, where 1 is the point $(1,0)$. Then ρ_θ acts on the vertices as the n-cycle $\sigma_n = (1,2,\dots,n)$. The permutation induced by r depends on the parity of n: if $n = 2m$ then r is the product of the $m-1$ transpositions $\tau_{2m} = (2,n)(3,n-1)\dots(m-1,m+1)$, if $n = 2m+1$, then r acts as the permutation $\tau_{2m+1} = (2,n)\dots(m,m+1)$. With this in mind we define the map

$$\begin{array}{rccc} \rho: & D_n & \to & S_n \\ & \rho_\theta & \mapsto & \sigma_n \\ & r & \mapsto & \tau_n. \end{array}$$

   It is easy to show that this gives an embedding of D_n in S_n: since permutations with support in different sets commute, and since the order of an k-cicle is k, σ_n has order n and both τ_{2m} and τ_{2m+1} have order two. Moreover an easy computation gives $\tau_n\sigma_n\tau_n = (1,m,m-1,\dots 2) = \sigma^{-1}$ hence the two permutation generate a dihedral group.

2. Let us consider the subgroup $H = \{1, x^5\}$ in the dihedral group D_{10}.

   (a) Prove that H is normal in D_{10} and that D_{10}/H is isomorphic to D_5.
   Solution We will show that H is contained in the center of D_{10}. This is equivalent to saying that $ghg^{-1} = h$ for every element h in H, and implies that H is normal. Clearly the identity 1 belongs to the center, hence it is enough to show that x^5 commutes with any element in G. Since x^5 is contained in the cyclic group generated by x, and the latter group is abelian, x^5 commutes with x. Moreover $yx^5y^{-1} = x^{-5}yy = x^5$. This implies that x^5 commutes with the generators of D_{10} hence it is in the center of D_{10}.
   In order to show that D_{10}/H is isomorphic to D_5, let us denote by π the projection map $\pi : D_{10} \to D_{10}/H = G$. The group G is generated by $\pi(x)$ and $\pi(y)$, the element $\pi(x)$ satisfies $\pi(x)^5 = \pi(x^5) = 1$, the element $\pi(y)$ satisfies $\pi(y)^2 = 1$, moreover the relation $\pi(x)\pi(y) = \pi(y)\pi(x)^{-1}$ is satisfied. This implies that there is a surjective homomorphism $D_5 \to D_{10}/H = G$. Since the two groups have the same cardinality they are isomorphic.

(b) Is D_{10} isomorphic to $D_5 \times H$?
Solution The two groups are isomorphic: indeed let us consider the subgroup K of D_{10} generated by x^2 and y. The element x^2 has clearly order 5, moreover $x^2y = yx^{-2} = y(x^2)^{-1}$. This implies that K is isomorphic to D_5. Moreover K is normal since K has index 2 in D_{10}. Since the intersection $H \cap K$ is trivial, D_{10} is isomorphic to $D_5 \times H$.

3. Determine the center of the dihedral group D_n.
Solution Recall that the center of a group is the subgroup consisting of all the elements that commute with every other element in the group.

The group D_1 is $\mathbb{Z}/2\mathbb{Z}$, and D_2 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Both are abelian, hence the center coincides with the whole group D_i.

Let us now assume that $n > 2$. We first show that no reflection is in the center: let us assume by contradiction that the element yx^k is in the center, in particular yx^k must commute with x and we get $xyx^kx^{-1} = yx^{k-2} = yx^k$. This implies that 2=0 modulo n which is a contradiction.

We now know that an element in the center has the form x^k for some k. Clearly any element of the form x^k commutes with x. Hence it is enough to check if the element commutes with y. But since $yx^ky = x^{-k}$, the element x^k is in the center if and only if $2k = 0$ modulo n. Hence there are two cases

(a) n is odd: the center of D_{2m+1} is $\{e\}$.
(b) n is even: the center of D_{2m} is $\{e, x^m\}$.

4. What is the smallest n such that

(a) $\mathbb{Z}/p_1p_2\mathbb{Z}$ embeds into S_n where p_1 and p_2 are two distinct primes? *Hint:* $\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.
Solution Any permutation can be written in a unique way as a product of cycles with disjoint support. We will denote this decomposition as the reduced cyclic decomposition of the permutation σ. Since permutations with disjoint support commute, the order of a permutation is the minimum common multiple of the lengths of the cycles in the reduced cyclic decomposition.
The smallest n such that $\mathbb{Z}/p_1p_2\mathbb{Z}$ embeds in S_n is $p_1 + p_2$. Let us first show that $\mathbb{Z}/p_1p_2\mathbb{Z}$ embeds in $S_{p_1+p_2}$. We know that $\mathbb{Z}/p_1p_2\mathbb{Z}$ as $\mathbb{Z}/p_1\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z}$ since p_1 generates a cyclic subgroup H of $\mathbb{Z}/p_1p_2\mathbb{Z}$ of order p_2, and p_2 generates a cyclic subgroup K of $\mathbb{Z}/p_1p_2\mathbb{Z}$ of order p_1 and $H \cap K = \{0\}$.
Let us denote by a the generator of $\mathbb{Z}/p_1\mathbb{Z}$ and by b the generator of $\mathbb{Z}/p_2\mathbb{Z}$. Let us consider the permutations $\alpha = (1 \dots p_1)$ and $\beta = (p_1 + 1, \dots\ p_1 + p_2)$. α has order p_1, β has order p_2 and they commute since they have disjoint support. This shows that there is an embedding of $\mathbb{Z}/p_1p_2\mathbb{Z}$ in $S_{p_1+p_2}$.
We want to show that there are no embeddings of $\mathbb{Z}/p_1p_2\mathbb{Z}$ in S_n with $n < p_1 + p_2$. Indeed assume by contradiction that this happens. We can assume

without loss of generality that $p_1 > p_2$. Let us denote again by α and β the images of a and b in S_n. The cyclic decomposition of α has a single p_1 cycle since $p_1 + p_2 < 2p_1$. Moreover since b has order p_2, the reduced cyclic decomposition of β consists of some disjoint p_2-cycles. The β conjugate of α is a p_1 cycle whose elements are relabelled according to the permutation β. In particular, since $n < p_1+p_2$, the supports of α and β cannot be disjoint, hence the permutations α and β don't commute, and this gives a contradiction.

(b) The dihedral group D_{15} embeds into S_n? *Hint:* write D_{15} as a semidirect product.
Solution Since the cyclic group $\mathbb{Z}/15\mathbb{Z}$ is a subgroup of D_{15}, n must be bigger or equal to 8. We now show that 8 works. Indeed let us consider the permutations $\sigma = (1,2,3,4,5)(6,7,8)$ and $\tau = (2,5)(3,4)(7,8)$. σ has order 15, τ has order 3, for the composition we have

$$\tau\sigma\tau = (2,5)(3,4)(7,8)(1,2,3,4,5)(6,7,8)(2,5)(3,4)(7,8) = \\ (1,5,4,3,2)(6,8,7) = \sigma^{-1}.$$

In particular the permutations σ and τ generate a subgroup of S_{15} that is isomorphic to D_{15}.

5. What is the subgroup of O_2 generated by two reflections in $\mathbb{R}^2$ about two lines of angle $\frac{2\pi}{n}$?
Solution Let us denote by H the subgroup generated by the two reflections and let us denote by l_1 and l_2 the two lines and by r_1 and r_2 the reflections along each of those. Let us moreover consider the regular n-gon Δ with center the origin and having a vertex on the line l_1 and a vertex on the line l_2. Since both reflections preserve the n-gon, so does the group generated by them. This implies that H is a finite subgroup of O_2 and hence is either a cyclic group or a dihedral group. Since H contains two different reflections, it is not a cyclic group, and hence it is a dihedral group.

In order to determine which is the order of H, let us recall that the reflections r_1 and r_2 have expression $\rho_{2\alpha}r$ and $\rho_{2\alpha+2\theta}r$ where α is the angle made by a line with the horizontal axis and $\theta = \frac{2\pi}{n}$. In particular the composition r_1r_2 is a reflection of angle $\frac{4\pi}{n}$. In particular the group H is the subgroup of D_n generated by a reflection and by $\rho_{2\theta}$. Now $\rho_{2\theta}$ generates C_n if n is odd and an index 2 subgroup C_m if n is even. This shows that there are two cases

(a) n is odd: H is isomorphic to D_n

(b) $n = 2m$ is even: H is isomorphic to D_m.

6. Let G be a discrete group of isometry of the plane in which every element is orientation preserving. Prove that the point group $\overline{G}$ is a cyclic group of rotations and that there is a point p in the plane such that the set of group elements which

fix p is isomorphic to $\overline{G}$. Restate this result in term of short exact sequences. *Hint* Recall that any orientation preserving isometry of the plane is a rotation about some point of the plane p.
Solution Since every element g of G is orientation preserving, it can be written as $t_a\rho_\theta$, where we denote by t_a the translation of the vector a and by ρ_θ the rotation of angle θ. Since the point group $\overline{G}$ is the image of G in O_2 under the isomorphism $\pi : M_2 \to O_2 = M_2/T_2$, the point group $\overline{G}$ contain only rotations. In particular by the classification of finite subgroups of O_2, it is a cyclic group.

Let us now consider a generator $\overline{g}$ of $\overline{G}$ and an element g in G such that $\pi(g) = \overline{g}$. Since g is an orientation preserving isometry of the plane, it has the form $t_a\rho_\theta$ for some $a \in \mathbb{R}^2$ and some angle θ. In particular there exists a point p such that g is a rotation through the angle θ about the point p. The subgroup of M_2 generated by g is a cyclic group of rotations around the point p that is isomorphic to $\overline{G}$.

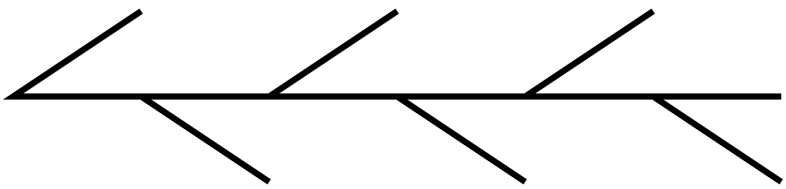
We only need to show that $H = \langle g \rangle$ consists precisely of the set of group elements which fix p, but, since every element of G is orientation preserving, the only isometries of G that fix p are rotations around p and the group H already contains all the rotations around p that are contained in G since it contains the rotations of all possible angles that belong to $\overline{G}$.

Since $\overline{G}$ can be realized as a subgroup of G, the sequence

$$1 \longrightarrow L_G \longrightarrow G \xrightarrow{\pi|_G} \overline{G} \longrightarrow 1$$

is a split exact sequence.

7. Let us consider the pattern $\mathcal{P}$ in $\mathbb{R}^2$ of which a portion is drawn in the picture, and that we assume that is invariant under the subgroup of translations of vectors in $\mathbb{Z}e_1$. Let G be the group of isometries of $\mathcal{P}$. Prove that the sequence



$$1 \longrightarrow L_G \longrightarrow G \xrightarrow{\pi|_G} \overline{G} \longrightarrow 1$$

doesnt't split.
Solution Let us assume (up to choosing cohordinates) that $\mathbb{Z}2e_1$ is the translation group of G. And let τ to be the isometry of $\mathbb{R}^2$ that is the composition $t_{e_1}r$ where we denote by r the reflection along the orizontal axis.

Let us check that the group generated by τ and t_{2e_1} is the group of isometries of $\mathcal{P}$. Indeed no element of O_2 belongs to G, since no reflection and no rotation stabilizes the pattern. Moreover, if a is an element in G, the image $a(0)$ must be a vertex of the pattern. If $a(0)$ is even then $t_{-a(0)}a$ is an isometry of the pattern that fixes the origin, hence it must be the identity. In particular $a = t_a(0)$. A similar argument implies that if $a(0)$ is odd, $a = t_{a(0)-1}\tau$.

Since τ belongs to G, the point group $\overline{G}$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$ generated by a reflection in O_2. However there is no order two element in G, hence the sequence

$$1 \longrightarrow L_G \longrightarrow G \xrightarrow{\pi|_G} \overline{G} \longrightarrow 1$$

cannot split.

8. Let $G = GL_n(\mathbb{R})$ act on the set $V = \mathbb{R}^n$ by left multiplication.

   (a) Describe the decomposition of V into orbits for this action.
   Solution Every element g in G fixes the origin, since G acts on $\mathbb{R}^n$ via linear transformations. This implies that the point $\{0\}$ form an orbit for the $GL_n(\mathbb{R})$ action. Moreover for any nonzero vector v there exist an invertible matrix g such that $ge_1 = v$. This implies that the complement of 0 in $\mathbb{R}^n$ is the other orbit of the G action.

   (b) What is the stabilizer of e_1?
   Solution The stabilizer of e_1 consists of those invertible matrices g such that $ge_1 = e_1$ that is of all the invertible matrices such that the first column of g is the vector e_1.

9. The group M_2 of isometries of the plane acts on the set of lines in the plane. Determine the stabilizer of a line.
   Solution We will first show that the group M_2 acts transitively on the set of lines in the plane. Indeed let $\mathcal{L}$ be a line and let p be a point in $\mathcal{L}$. The line $t_{-p}\mathcal{L}$ is a line through the origin. Let θ be its slope. Then $t_{-p}\mathcal{L} = \rho_\theta\mathcal{L}_0$ hence $\mathcal{L} = t_p\rho_\theta\mathcal{L}_0$ where $\mathcal{L}_0$ is the horizontal line. This implies that it is enough to determine the stabilizer H of the line $\mathcal{L}_0$. The stabilizer of a line through the point p with slope θ will be the conjugate group $\rho_{-\theta}t_{-p}Ht_p\rho_\theta$.

   Let us now compute H. Since $\mathcal{L}_0$ contains the origin, the subgroup of H consisting of the translations that are contained in H coincides with the line itself. We will now show that the point group $\overline{H}$ in this case is the subgroup of O_2 that stabilizes the line, that coincides with the dihedral group D_2 that is isomorphic to the Klein four group. Indeed if an isometry $t_a\varphi$ belongs to H, then t_a belongs to H since $t_a\varphi(0) = t_a(0) = a$ hence a must belong to $\mathcal{L}_0$. This implies that if $t_a\varphi$ belongs to H also $\varphi = t_{-a}t_a\varphi$ must belong to H.

   This implies that there is a split exact sequence

$$1 \longrightarrow \mathbb{R} \longrightarrow H \longrightarrow D_2 \longrightarrow 1.$$

In particular the stabilizer of $\mathcal{L}_0$ splits as a product $\mathbb{R} \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. And is generated by the translations along the horizontal line, the reflection r and the rotation ρ_π.