Solution 9

Representations of finite groups, characters

1. Show that the image of a representation of dimension 1 of a finite group is a cyclic group.
Solution Let G be a finite group. A representation of dimension 1 of G is an homomorphism $\rho : G \to GL_1(\mathbb{C})$. The group $GL_1(\mathbb{C})$ coincides with the multiplicative group $\mathbb{C}^\times$. Let z be an element in the image of ρ. Since G is finite, we get that $z^n = 1$ for some n, in particular the image of ρ is a subgroup of the group S^1 of complex numbers of modulus one. We saw in class that all finite subgroups of S^1 are cyclic and this finishes the proof.

2. Let H be a subgroup of index 2 of a group G, and let $\sigma : H \to GL(V)$ be a representation. Let a be an element in G not in H. Define a *conjugate* representation $\sigma' : H \to GL(V)$ by the rule $\sigma'(h) = \sigma(a^{-1}ha)$. Prove that

   (a) σ' is a representation of H.
   SolutionIn order to check that σ' is well defined, let us notice that, since H has index 2 in G, H is normal, hence for every element $h \in H$ the element $a^{-1}ha$ still belongs to H, in particular the value $\sigma(a^{-1}ha)$ is well defined. Let us now check that σ' is a representation. In order to do this, it is enough to verify that σ' is an homomorphism in GL_n for some n. In particular, since σ is a representation in $GL(V)$, the image of σ' is also contained in $GL(V)$, hence we only have to verify that $\sigma'(gh) = \sigma'(g)\sigma'(h)$. But this follows from the definition:

   $$\sigma'(gh) = \sigma(a^{-1}gha) = \sigma((a^{-1}ga)(a^{-1}ha)) = \sigma(a^{-1}ga)\sigma(a^{-1}ha) = \sigma'(g)\sigma'(h).$$

   (b) If σ is the restriction to H of a representation of G then σ' is isomorphic to σ.
   Solution Assume that σ is the restriction of a representation of G, and let $A \in GL(V)$ be the element $\sigma(a)$. In order to show that σ and σ' are conjugate we need a linear isomorphism $L : V \to V$ such that, for any $h \in H$, $\sigma(h)(Lv) = L(\sigma'(h)v)$. The linear map A is such an isomorphism, indeed for any $v \in V$

   $$\sigma(h)Av = AA^{-1}\sigma(h)Av = A\sigma(a^{-1})\sigma(h)\sigma(a)v = A\sigma(a^{-1}ha) = A\sigma(h)v.$$

   (c) If b is another element of G not in H, then the representation $\sigma'' = \sigma(b^{-1}hb)$ is isomorphic to σ'.
   Solution Since b is another element of G that doesn't belong to H, and since

H has index 2 in G, then there exists an element h in H such that $b = ah$. Now we have, for every $g \in H$, that $\sigma''(g) = \sigma(h^{-1}a^{-1}gah) = \sigma(h^{-1})\sigma'(g)\sigma(h)$. In particular this implies that, for every $g \in H$, we have $\sigma(h)\sigma''(g) = \sigma'(g)\sigma(h)$ and the linear map $\sigma(h) : V \to V$ gives an isomorphism of the representations σ' and σ''.

3. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a finite group on a real vector space V. Prove the following:

   (a) There exists a G-invariant, positive definite, symmetric form $\langle , \rangle$ on V.
   Solution Let us fix a positive definite, symmetric bilinear form $[\ ,\ ]$ on V. (To find such a form it is enough to fix an isomorphism of V with $\mathbb{R}^n$ and consider the standard positive definite symmetric bilinear form on $\mathbb{R}^n$). And let us define the averaged form by setting

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)w].$$

   The form is symmetric, positive definite and G-invariant. The fact that is symmetric follows from the fact that $[\ ,\ ]$ is symmetric:

$$\langle v, w\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)w] = \frac{1}{|G|}\sum_{g\in G}[\rho(g)w, \rho(g)v] = \langle v, w\rangle.$$

   To check that the form is positive, it is enough to check that $\langle v, v\rangle > 0$, but we have

$$\langle v, v\rangle = \frac{1}{|G|}\sum_{g\in G}[\rho(g)v, \rho(g)v] > 0$$

   since the latter expression is a sum of positive numbers, the verification of the G invariance follows rearranging the summation, once one notices that for any element $h \in G$, the right multiplication by h gives a permutation of G:

$$\begin{aligned}\langle \rho(h)v, \rho(h)w\rangle &= \tfrac{1}{|G|}\textstyle\sum_{g\in G}[\rho(g)\rho(h)v, \rho(g)\rho(h)w]\\ &= \tfrac{1}{|G|}\textstyle\sum_{gh\in G}[\rho(gh)v, \rho(gh)w] = \langle v, w\rangle.\end{aligned}$$

   (b) ρ is a direct sum of irreducible representations.
   Solution If ρ is irreducible, there is nothing to prove. Assume that ρ is not irreducible and let $W < V$ a ρ-invariant subspace. We claim that the orthogonal of W with respect to the form $\langle\ ,\ \rangle$ defined in the previous part is also ρ-invariant. Indeed it is enough to check that if $z \in W^\perp$ and $g \in G$, then $\rho(g)z$ is in $W^\perp$, or equivalently for every $w \in W$ $\langle \rho(g)z, w\rangle = 0$. But this is true since

$$\langle \rho(g)z, w\rangle = \langle \rho(g^{-1})\rho(g)z, \rho(g^{-1})w\rangle = \langle z, \rho(g^{-1})w\rangle = 0$$

Here the first equality is due to the fact that $\langle\ ,\ \rangle$ is ρ-invariant, the second one to the fact that ρ is a representation, the third to the fact that $\rho(g^{-1})w \in W$ since W is G-invariant, and z belongs to $W^{\perp}$. This implies that ρ splits as a direct sum of two representations ρ', ρ''. Since V is finite dimensional the conclusion follows by induction.

4. Let us consider the representation ρ of $\mathbb{Z}$ on $\mathbb{C}^2$ defined by $\rho(1) = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$.

   (a) Find a proper invariant subspace.
   Solution Clearly the subspace of $\mathbb{C}^2$ generated by the first element of the standard basis is invariant under the representation ρ: indeed any element in $\langle e_1 \rangle$ has expression $\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$ for some $a \in \mathbb{C}$, and for any $n \in \mathbb{Z}$, we have that $\rho(n)$ is the matrix $\left[\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right]$ and we have $\left[\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right]$.

   (b) Show that ρ is not the direct sum of irreducible representations.
   Solution Assume by contradiction that ρ is the direct sum of irreducible representations. Then there would exist another vector $w \in \mathbb{C}^2$ that doesn't belong to $\langle e_1 \rangle$, and such that, for every $n \in \mathbb{Z}$, $\rho(n)w = w$. But since already $\rho(1)$ has no other eigenvector apart from e_1 this is clearly not possible.

5. Determine the character table for the Klein four group.
   Solution The Klein four group K is isomorphic to the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since it is abelian its conjugacy classes coincides with the four group elements. Let us call a and b the generators of K. In order to compute the character table for K it is enough to determine four distinct non isomorphic one dimensional representations of K. If ρ is a one dimensional representation of K, the image of a generators of K should be an element of order two in $\mathbb{C}^{\times}$, in particular it should be an element of the set $\{\pm 1\}$. Moreover for any choice we get a non isomorphic representation of K. This implies that the character table of K is

K	1	a	b	ab
ρ_1	1	1	1	1
ρ_a	1	-1	1	-1
ρ_b	1	1	-1	-1
ρ_{ab}	1	-1	-1	1

6. Let us consider the Dihedral group D_5, and its cyclic subgroup C_5.

   (a) Determine the character table of D_5 and of C_5.
   Solution Let ω denote a primitive fifth root of unit in $\mathbb{C}^*$ that is a number such that $\omega^5 = 1$. There are five conjugacy classes in C_5, corresponding to the five elements. Moreover in order to determine a representation of C_5 it is enough to describe the image of the generator that is going to be a fifth root of unity, hence a power of ω. In particular we get that the character table of $\mathbb{C}^5$ is

C_5	0	1	2	3	4
τ_1	1	1	1	1	1
τ_ω	1	ω	ω^2	ω^3	ω^4
τ_{ω^2}	1	ω^2	ω^4	ω	ω^3
τ_{ω^3}	1	ω^3	ω	ω^4	ω^2
τ_{ω^4}	1	ω^4	ω^3	ω^2	ω

In order to compute the conjugacy classes in D_5 let us notice that, since 5 is odd, all reflections are conjugate, hence form a conjugacy class C_y, moreover the rotations come in three different conjugacy classes: $\{1\}, \{x, x^4\}, \{x^2, x^3\}$. This implies that we have to exhibit five different irreducible representations of D_5. We will denote by ρ_1 the trivial representation. Let us consider the subgroup C_5 of D_5. It is a normal subgroup and the quotient $D_5/C_5 = \mathbb{Z}/2\mathbb{Z}$. This gives another one dimensional (hence irreducible) representation of D_5, the sign representation. We will denote it by sign.

Let us now consider the standard representation of D_5 as a subgroup of $O_2(\mathbb{R})$. We showed in class that the group D_5 is isomorphic to the subgroup of O_2 generated by a reflection and a rotation of angle $2\pi/5$. The matrix expression for a rotation of angle $2\pi/5$ is $R_x = \begin{bmatrix} \cos(2\pi/5) & \sin(2\pi/5) \\ -\sin(2\pi/5) & \cos(2\pi/5) \end{bmatrix}$ and the matrix expression for the reflection along the x axis is $R_y = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Interpreting these two matrices as complex matrices, and letting them act on $\mathbb{C}^2$ we get a complex representation of D_5 that we will denote by ρ_ω. It is well known that if ω can be chosen to be $\cos(2\pi/5) + i\sin(2\pi/5)$. In particular it is easy to compute the character of ρ_ω (see the table below). Since the character has norm one, we get that the representation is irreducible. The last irreducible representation $\rho_{\omega^2}$of D_5 is obtained in a similar manner, by defining $\rho_{\omega^2}(y) = R_y$ and $\rho_{\omega^2}(x) = R_x^2$. Computing the character of this representation one can easily check that ρ_{ω^2} is irreducible and ρ_{ω^2} is not isomorphic to ρ_ω. This leads to the character table for D_5.

D_5	$\{1\}$	$\{x, x^4\}$	$\{x^2, x^3\}$	C_y
ρ_1	1	1	1	1
sign	1	1	1	-1
ρ_ω	2	$\omega + \overline{\omega}$	$\omega^2 + \overline{\omega}^2$	0
ρ_{ω^2}	2	$\omega^2 + \overline{\omega}^2$	$\omega + \overline{\omega}$	0

(b) Decompose the restriction of each irreducible character of D_5 into irreducible characters of C_5.

Solution The restriction to C_5 of the characters of ρ_1 and sign equal to the trivial character, the character of ρ_ω is the sum of the characters of τ_ω and τ_{ω^4}, in a similar way ρ_{ω^2} is the direct sum of τ_{ω^2} and τ_{ω^3}.

7. The Quaternion group Q is the group $Q = \langle i, j, k | i^2 = j^2 = k^2 = -1, ijk = -1 \rangle$

(a) Find a subgroup of $GL_2(\mathbb{C})$ isomorphic to Q, determine the order of Q.
Solution Let us consider the elements of $GL_2(\mathbb{C})$ $I = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $K = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]$. We have $I^2 = -\mathrm{Id}$, $J^2 = -\mathrm{Id}$, $K^2 = -\mathrm{Id}$, moreover $IJK = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right] = -\mathrm{Id}$. In particular the subgroup Q can be realized as a subgroup of $GL_2(\mathbb{C})$ and has eight elements: $\{\pm\mathrm{Id}, \pm I, \pm J, \pm K\}$.

(b) Determine the conjugacy classes of Q.
Solution We will now identify Q with the subgroup of $GL_2(\mathbb{C})$ we just defined, to make explicit computations. Since the matrices $\pm Id$ commute with every matrix in $GL_2(\mathbb{C})$, in particular they commute with the elements in Q, hence they are in the center of Q. Moreover, from the fact that $I^2 = -1$ we get that $I^{-1} = -I$. We can now compute the relation $IJI^{-1} = -\left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right] = -J$. Analogously one gets that $KJK^{-1} = -J$, in particular the conjugacy class of J contains the two elements $\pm J$. In the same way one checks that the conjugacy class of I contains $\pm I$ and the conjugacy class of K contains $\pm K$.

(c) Prove that any subgroup of Q is normal.
Solution The computation above shows that the subgroup generated by I is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and is normal and the same is true for $\langle J \rangle$ and $\langle K \rangle$. The only other non trivial subgroup is the center $\{\pm 1\}$ of Q and clearly the center is normal.

(d) Write the character table of Q.
Solution In order to compute the character table we need to find 5 irreducible representations of Q, since there are 5 conjugacy classes. Of course there is the trivial representation, that we will denote by ρ_1. Moreover we saw above that the subgroup generated by I is normal. Since the quotient Q/I is $\mathbb{Z}/2\mathbb{Z}$ we get a representation ρ_i obtained by composing the sign representation of $\mathbb{Z}/2\mathbb{Z}$ with the quotient map. In the very same way (quotienting the subgroup generated respectively by J and K) one gets the representations ρ_j and ρ_k. All the representations ρ_1, ρ_i, ρ_j and ρ_k are one dimensional, hence irreducible, moreover computing the characters it is easy to see that they are not isomorphic, since the characters are different. We realized Q as a subgroup of $GL_2(\mathbb{C})$. This gives a two dimensional representation ρ of Q. That is irreducible since the eigenspaces of I and J are distinct. We can sum up the results we just obtained in the character table for Q:

Q	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
ρ_1	1	1	1	1	1
ρ_i	1	1	1	-1	-1
ρ_j	1	1	-1	1	-1
ρ_k	1	1	-1	-1	1
ρ	2	-2	0	0	0