D-MATH
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Algebra I

HS 2013

Test 2

Group theory

This test is long. Don't worry if you don't have time to solve it completely. In general it is better to solve some problems correctly than to write something on many problems.

1. In each of the following cases give an example, or say that no such example exists.

   (a) A p-group that is not abelian.

   (b) A group whose commutator subgroup is trivial.

   (c) An infinite group that is finitely presented.

   (d) A representation of a group that is not a direct sum of irreducible representations.

   (e) A simple group that is not cyclic.

   (f) A subgroup of a free group on two generators that is isomorphic to a free group on three generators.

   (g) Groups A, B, C such that $A \lhd B \lhd C$, but A is not normal in C.

   (h) A short exact sequence that does not split.

   (i) An subgroup of S_5 isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

   (j) Two subgroups of S_6 of order 4 that are not conjugate.

2. Let G be a group. A subgroup H of G is *characteristic* if $\varphi(H) = H$ for any automorphism φ of G.

   (a) Prove that the center $Z(G)$ is a characteristic subgroup of G.

   (b) Prove that any characteristic subgroup of G is in particular a normal subgroup. (*Hint*: consider the action of a group on itself by conjugation.)

   (c) Give an example of a normal subgroup of the quaternion group that is *not* characteristic.

3. Let a group G act transitively on a set X.

   (a) Show that for every pair of points x, y, the stabilizers G_x, G_y have the same cardinality.

(b) Deduce from the counting formula that

$$|G| = \sum_{x \in X} |G_x| = \sum_{g \in G} \mathrm{Fix}(g).$$

(c) Conclude from the previous formula that there exists an element of G acting without fixpoint.

[This implies that a group is not the union of the conjugacy classes of one of its subgroups.]

4. Let $p < q$ be primes. Show that a non-abelian group G of order pq may be written as a semidirect product. *Hint:* Sylow theory might help.

5. Let G be a non-abelian group of order 55. You know from exercise 4 that G is a semidirect product, and you can assume that a non-abelian semidirect product $\mathbb{Z}/5\mathbb{Z} \ltimes \mathbb{Z}/11\mathbb{Z}$ exists.

   (a) Determine the conjugacy classes and write the class equation of G.

   (b) The character table of $\mathbb{Z}/5\mathbb{Z}$ is

	$C(0)$	$C(1)$	$C(2)$	$C(3)$	$C(4)$
1	1	1	1	1	1
χ_1	1	ξ	ξ^2	ξ^3	ξ^4
χ_2	1	ξ^2	ξ^4	ξ^3	ξ
χ_3	1	ξ^3	ξ	ξ^4	ξ^2
χ_4	1	ξ^4	ξ^3	ξ^2	ξ

   Use this information to fill in 5 rows of the character table of G.

   (c) How many other irreducible representations are there? Determine their dimensions.

[Note that part (b) can be applied with any semidirect product.]