1.1.3 Gravimetry

25-3-2013

→ Deconvolution problem $Tf = g$

$$(Tf)(x) = \int_{\mathbb{R}} k(x-y) f(y)\, dy = (k * f)(x)$$

with kernel $k \in L^1(\mathbb{R})$

$$\Rightarrow \quad \|Tf\|_{L^2(\mathbb{R})} \le \|k\|_{L^1(\mathbb{R})} \|f\|_{L^2(\mathbb{R})}$$

By convolution thm. for Fourier transform

$$\mathcal{F}(Tf) = \underbrace{\mathcal{F}(k)}_{\mathcal{F}: L^1(\mathbb{R}) \to C^0(\mathbb{R})} \cdot \underbrace{\mathcal{F}(f)}_{\mathcal{F}: L^2(\mathbb{R}) \to L^2(\mathbb{R}) \text{ isometry}}$$

Recall Lebesgue lemma:

$$k \in L^1(\mathbb{R}) \implies \lim_{|\xi| \to \infty} \mathcal{F}(k)(\xi) = 0$$

Proof: Use that $C_0^\infty(\mathbb{R})$ (compactly supported) is dense in $L^1(\mathbb{R})$

Find $\phi \in C_0^\infty(\mathbb{R}) : \|\phi - k\|_{L^1(\mathbb{R})} \le \varepsilon$

Elementary: $\mathcal{F}(\phi)(\xi) = O(|\xi|^{-1})$ for $|\xi| \to \infty$

(from Fourier transform formula integrating by parts)

$$|\mathcal{F}(k)(\xi)| \le |(\mathcal{F}(k) - \mathcal{F}(\phi))(\xi)| + |\mathcal{F}(\phi)(\xi)| \le \varepsilon + \varepsilon \quad \text{for } |\xi| \text{ sufficiently large}$$

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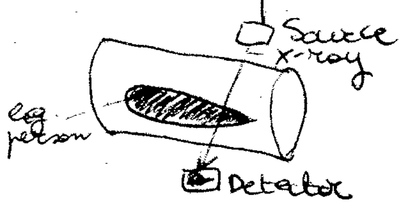
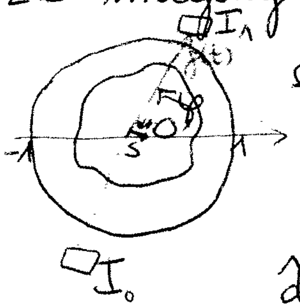
⇒ All deconvolution problems with L^1-kernel are ill-posed.

1.1.4 Computerized Tomography (CT)



2D imaging setup



$\Omega = B_1(0) = \{x \in \mathbb{R}^2 : \|x\| \le 1\}$

$f: \Omega \to \mathbb{R}$ density

$\operatorname{supp} f \subset \Omega$

$\hat{d}(\varphi) \hat{=}$ unit vector with angle φ

$$\gamma_{s\varphi}(t) = s\,\hat{d}(\varphi) + \tau\,\hat{d}(\varphi)^\perp$$

Reasonable hyp. the denser the material, the stronger the absorption.
Attenuation law for intensity I

$$\frac{d}{d\tau} I(\gamma(t)) = -f(\gamma(\tau)) \cdot I(\gamma(t))$$

$$\frac{d}{d\tau} \log(I(\gamma(t))) = -f(\gamma(t))$$

$$\int_{-1}^{1} \qquad \underbrace{-\log\left(\frac{I_1}{I_0}\right)}_{\text{Data } g(s,\varphi)} = \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(\gamma(\tau))\,d\tau$$

unknown $f = f(x)$

$$T_R f = g$$

for $\displaystyle (T_R f)(s,\varphi) := \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s\hat{d}(\varphi) + \tau\, \hat{d}(\varphi)^{\perp})\,d\tau$

Radon transform

<u>Theorem 1.4.A</u>

$T_R : L^2(\Omega) \longrightarrow L^2(]-1,1[\times [0,2\pi[)$
is bounded with norm $\frac{4}{\pi}$.

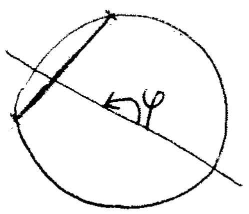
<u>Proof</u>:

$$|T_R f(s,\varphi)|^2 \overset{C.S.}{\leq} 2\sqrt{1-s^2} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} |f(\gamma_{s,\varphi}(\tau))|^2 d\tau$$

$$\|T_R f\|_{L^2}^2 = \int_{-1}^{1}\int_{0}^{2\pi} |T_R f(s,\varphi)|^2 d\varphi\, ds \leq$$

$$\leq 2 \int_0^{2\pi} \Big\{ \underbrace{\int_{-1}^{1} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} |f(\gamma_{s,\varphi}(\tau))|^2 d\tau\, ds}_{=\text{integration over the all disc no matter the } \varphi} \Big\} d\varphi =$$

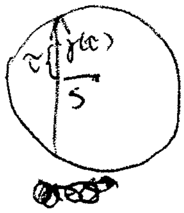
$$= 4\pi \int_{\Omega} |f(x)|^2 dx \qquad \blacksquare$$



Simpler situation: $f(x) = f(|x|) \implies g = g(s)$
(rotational symmetry) (no dependence on φ)

Substitution: $f(x) = \phi(\|x\|^2)$

$$\Rightarrow T_R f(s,0) = 2\int_0^{\sqrt{1-s^2}} \phi(s^2+\tau^2)\, d\tau =$$



$$\underset{r = s^2+\tau^2}{=} 2\int_{s^2}^{1} \phi(r)\frac{1}{2\sqrt{r-s^2}}\, dr = g(s) =$$

$dr = 2\tau\, d\tau$

$d\tau = \frac{dr}{2\tau} = \frac{dr}{2\sqrt{r-s^2}}$

$$T_A(\phi(x)) \underset{x=s^2}{=} \int_x^1 \frac{\phi(r)}{\sqrt{r-x}}\, dr = g(\sqrt{x}) = \tilde{g}(x) \qquad 0 \le x \le 1$$

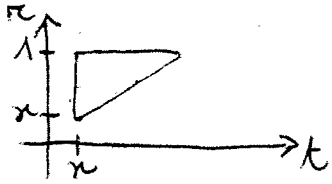
again convolution operator restricted to a finite interval

$\widehat{=}$ Convolution operator with kernel

$$k(\sigma) = \begin{cases} 0 & , \sigma < 0 \text{ or } \sigma > 1 \\ \frac{1}{\sqrt{\sigma}} & , 0 \le \sigma \le 1 \end{cases} \qquad \in L^1(\mathbb{R})$$

$\rightarrow$ known as **Abel** integral operator

$$(T_A^2\phi)(x) = \int_x^1 \int_t^1 \frac{\phi(r)}{\sqrt{r-t}\,\sqrt{t-x}}\, dr\, dt =$$



$$= \int_x^1 \Bigg(\underbrace{\int_x^r \frac{1}{\sqrt{r-t}\,\sqrt{t-x}}\, dt}_{= \int_{-\alpha}^{\alpha} \frac{1}{\sqrt{\alpha^2 - r^2}} dr \,=\, \pi \text{ ("arcsin integral")}}\Bigg) \phi(r)\, dr =$$

$$= \pi \int_x^1 \phi(r)\, dr$$

$$\Rightarrow \frac{d}{dx}(T_A^2\phi)(x) = -\pi\phi(x)$$

"$T_A \approx \sqrt{\frac{d}{dx}}^{-1}$"

Abel integral equation

$T_A\phi = \tilde{g}$

$\Rightarrow T_A^2\phi = T_A\tilde{g}$

$\overset{\frac{d}{dx}}{\Rightarrow} \phi = -\frac{1}{\pi}\frac{d}{dx} T_A \tilde{g}$

ill-posed since it has (see exam

11.5 Backward heat equation

$u = u(x,t)$ temperature, $0 \le x \le 1$
$0 \le t \le t^*$

$$\begin{cases} \partial_t u = \partial_x^2 u & \text{on }]0,1[\times]0,t^*[\\ u(0,t) = u(1,t) = 0 & 0 \le t \le t^* \\ u(x,0) = f(x) & 0 \le x \le 1 \end{cases}$$

Data: $g(x) = u(x,t^*)$
unknown $f(x)$

Separation of variables (Fourier series)

$$f(x) = \sum_{n=1}^{\infty} f_n \underbrace{\sqrt{2}\sin(\pi n x)}_{\text{ONB of } L^2(]0,1[)} \qquad \text{for } f \in L^2(]0,1[)$$

Γ separation of variables $u(x,t) = \sum_n c_n e^{-\pi^2 n^2 t}\sqrt{2}\sin(\pi n x)$.

$$\Rightarrow \text{solution} \quad u(x,t) = \sum_{n=1}^{\infty} f_n e^{-\pi^2 n^2 t}\sqrt{2}\sin(\pi n x) =$$

$$f_n = \int_0^1 f(x)\sqrt{2}\sin(n\pi x)\,dx$$

$$= \sum_{n=1}^{\infty}\int_0^1 f(y)\sqrt{2}\sin(n\pi y)e^{-\pi^2 n^2 t}\sqrt{2}\sin(\pi n x)\,dy$$

At t^* ~~$u(x, t^*)$~~ $e^{-\pi n^2 t^*}$ converges very fast $\Rightarrow \sum_n$ conv. uniformly

$$\Rightarrow \sum\int = \int\sum$$

$$u(x,t^*) = \int_0^1 \underbrace{\left(\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} 2\sin(n\pi y)\sin(n\pi x)\right)}_{= k(x,y)} f(y)\,dy$$

$$= \int_0^1 k(x,y) f(y)\,dy$$

$$k(x,y) = \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \underbrace{2\sin(n\pi y)\sin(n\pi y)}_{\subset \text{ONB of } L^2(]0,1[^2)}$$

when $\sin(n\pi y)\sin(m\pi y)$ n, m independent

is analytic
due to exponential decay of Fourier coefficients.

Diagonalization of integral operator, because
$\sin(\pi l \cdot)$, $l \in \mathbb{N}$ are its eigenfunctions with eigenvalues

$$\int_0^1 k(x,y)\sin(\pi l y)\,dy = \underbrace{e^{-\pi^2 l^2 t^*}}_{\text{eigenvalue}} \sin(\pi l x)$$

$e^{-\pi^2 l^2 t^*}$ → superexponential decay to 0

$\Rightarrow$ $g(x) = \sin(\pi l x)$, $\|g\|_{L^2} = \frac{1}{\sqrt{2}}$

will induce $f(x) = e^{\pi^2 l^2 t^*} \sin(\pi l x)$

$\|f\|_{L^2} = \frac{1}{\sqrt{2}} e^{\pi^2 l^2 t^*} \longrightarrow \infty$ for $l \to \infty$

"extremely ill-posed"

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