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I.2. Approaches to reconstruction. A survey

$X, Y \hat{=}$ Hilbert spaces, $T \in \mathcal{L}(X, Y)$

Ill-posed operator equation : $Tf = g$

Noisy data : $\|g^\delta - g\| \le \underset{\text{noise level}}{\delta}$

2.1 Tikhonov regularization

Least squares approach to $Tf \approx g$

$$f \approx f_\delta^* \in \underset{f \in X}{\operatorname{argmin}} \underbrace{\|Tf - g^\delta\|_Y^2}_{\text{data misfit functional } J_0(f)} \quad (1.2.A)$$

~~Thes~~ least squares problem will be affected by ~~ill-posenes~~ ill-poser of the problem.

Idea: Remedy instability of (1.2.A) through adding a penalty term:

$$J_0(f) \longrightarrow J_\alpha(f) = J_0(f) + \alpha \|f - f_0\|_X^2$$

where $\alpha > 0$, $f_0 \in X$ some guess for f ($f_0 = 0$ possible)

$J_\alpha(f) \hat{=}$ quadratic functional

$$J_\alpha(f) = \frac{1}{2} \underset{\substack{\text{bilinear} \\ \text{form}}}{a(f,f)} - \underset{\substack{\text{linear} \\ \text{form}}}{\ell(f)} + \underset{\in \mathbb{R}}{c}$$

with
$$a(f, f') = 2(Tf, Tf')_Y + 2\alpha (f, f')_X$$
$$\ell(f) = 2(Tf, g^\delta)_Y + 2\alpha (f, f_0)_X$$

Note: $\underbrace{a(f,f) \ge 2\alpha \|f\|_X^2}_{a \text{ is } X\text{-elliptic}}, \ \forall f \in X$

$\Rightarrow J_\alpha$ strictly convex

$\Rightarrow \exists!$ minimizer to J_α

Theorem 2.1 C: $f \to J_\alpha(f)$ has a unique minimizer $f_\alpha^+ \in X$ for all $\alpha > 0$ that satisfies

$$a(f_\alpha^+, v) = \ell(v) \quad \forall v \in X \qquad (2.1 D)$$

Follows from: $J_\alpha(f) - J_\alpha(f_\alpha^+) = \frac{1}{2} a(f - f_\alpha^+, f - f_\alpha^+)$

$f = f_\alpha^+ \Leftrightarrow a(f - f_\alpha^+, f - f_\alpha^+) = 0$

$(2.1.D) \Rightarrow (Tf_\alpha^+, Tv)_Y + \alpha(f_\alpha^+, v)_X = (\cancel{T}v, g^\delta)_Y + \alpha(v, f_0)_X$ $\quad T^*$
$\forall v \in X$

[Adjoint operator: $T^*: Y \to X$: $(T^*w, v)_X = (w, Tv)_Y \quad \forall v \in X, w \in Y$
$\|T^*\| = \|T\|$]

$(2.1.D) \Rightarrow$ Operator equation in X:

$$(T^*T + \alpha\, \mathrm{Id}) f_\alpha^+ = T^* g^\delta + \alpha f_0 \quad (2.1E)$$

$$\Rightarrow R_\alpha g^\delta = f_\alpha^+ = (T^*T + \alpha\, \mathrm{Id})^{-1}(T^* g^\delta + \alpha f_0)$$

reconstruction by Tikhonov regularization

For $f_0 = 0$: $R_\alpha : Y \to X$ is linear
(solution operator for (2.1E)) with
$\|R_\alpha\| \le \frac{\|T^*\|}{\alpha}$

$$\|R_\alpha w\|_X^2 \le \frac{1}{2\alpha} a(R_\alpha w, R_\alpha w) \overset{(2.1D)}{\le} \frac{1}{2\alpha} \ell(R_\alpha w) \le$$
$w \in Y$
$$\le \frac{1}{\cancel{2}\alpha}(T^* w, R_\alpha w)_X \overset{CS}{\le} \frac{\|T^*\|}{\alpha} \|R_\alpha w\|_X \|w\|_Y$$

We expect
α large $\Rightarrow$ large regularization error
α small $\Rightarrow$ large impact of noise (($\|R_\alpha\|$ bigger and bigger)
$\Rightarrow$ key point: choice of α

~~$J_\alpha(f)$~~

Generalization of Tikhonov regularization

$$J_\alpha(f) \longrightarrow J_0(f) + \alpha R(f)$$

(possibly non quadratic) regularization term

Quadratic choice: with $L: X \to Z$: $R(f) = \|Lf\|_Z$

2.2 Iterative regularization

Idea: Apply an iterative minimization procedure to (1.2 A)
Simplest choice: linear descent

$$J_0(f) = \frac{1}{2}\|Tf - g^\delta\|_Y^2 \to \min$$

$$(\mathrm{grad}\, J_0(f), v)_X = (Tf - g^\delta, Tv)_X$$

chain rule

Riesz representative of the derivative

$$= T^*(Tf - g^\delta)$$

Initial guess: $f_0^+ \in X$

$$f_{k+1}^+ = f_k^+ - \mu T^*(Tf_k^+ - g^\delta) \quad (2.2c)$$

with stepsize parameter $\mu > 0$

Sufficient for convergence:

$$\phi(v) - \phi(w) = (Id - \mu T^* T)(v - w)$$

$\Rightarrow$ ~~a~~ contraction if $\|Id - \mu T^* T\| < 1$

if $\mu < \frac{2}{\|T^* T\|}$

~~bea~~ because T^*T is non-negative and selfadjoint

(2.2c) is called Landweber iteration.

Associated reconstruction operator:

$$R_k g^\delta = f_k^+ \quad , k \in \mathbb{N}$$

If $f_0 = 0 \Rightarrow R_k : X \to Y$ is linear and bounded:

because $R_k w = \sum_{j=0}^{k} (Id - \mu T^* T)^j T^* w$

and it can be proved by induction.

We expect: k small $\Rightarrow$ large iteration error

k large $\Rightarrow$ big impact of noise

because we get closer to T^{-1}, which is unbounded.