9-10-2013

I.3 Reconstruction and Regularization

1.3.1 Generalized Inverses

X, Y Hilbert spaces, $T \in \mathcal{L}(X,Y)$

Generalized inverse (pseudo-inverse) is a linear operator
$T^+ : Y \to X$ defined through

(i) $T^+g \in \underset{\tilde f \in X}{\operatorname{argmin}} \|T\tilde f - g\|_Y := \{\tilde f \in X : \|T\tilde f - g\|_Y = \inf_{v \in X} \|Tv - g\|_Y\}$

(ii) $\|T^+g\|_X$ minimal in argmin

T linear $\Rightarrow$ argmin convex $\Rightarrow \exists$ minimizer <u>unless argmin $= \emptyset$</u>

argmin $= \emptyset$ is possible, e.g. for $g \in \overline{R(T)} \setminus R(T)$

$\Rightarrow$ if $R(T)$ is not closed, T^+ is not defined on all of Y:
defined only on domain $\mathcal{D}(T^+) \subset Y$.

Assume $g \in \mathcal{D}(T^+)$

$T^+g \in$ argmin $\ldots \neq \emptyset \Rightarrow (TT^+g - g, Tv)_Y = 0 \quad \forall v \in X$

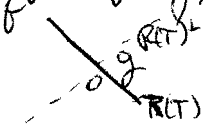
$\Leftrightarrow \boxed{T^*T(T^+g) = T^*g}$ (Normal equations)

<u>Lemma 1.3.1 E</u> $\quad \mathcal{D}(T^+) = R(T) + R(T)^\perp$

[$M^\perp \hat{=}$ orthogonal complement]

<u>Proof</u> "$\subset$": $g \in R(T) + R(T)^\perp \quad g = Tf + g^\perp$ for some $f \in X$

$g \in R(T)^\perp \quad \inf_{\tilde f} \|T\tilde f - g\|_Y \Rightarrow 0 \in$ argmin $\Rightarrow$ argmin $\neq \emptyset$



$g \in R(T)$: $\quad g = Tf \Rightarrow f \in$ argmin $\Rightarrow$ argmin $\neq \emptyset$

$\Rightarrow g \in \mathcal{D}(T^+)$

"$\supset$": $g \in \mathcal{D}(T^+) \Rightarrow T^*(T(T^+g) - g) = 0$ (normal equatio

$\underbrace{T(T^+g)}_{\in R(T)} - g \in \cancel{\operatorname{Ker}(T^*)} \boxed{\mathcal{N}(T^*) \overset{\text{thm. for bounded operators in Hilbert spaces}}{=} R(T)^\perp}$

$\Rightarrow g \in R(T)^\perp + R(T)$ $\blacksquare$

$\Rightarrow \mathcal{D}(T^+) \subset Y$ dense

Lemma 1.3.1F $\quad g \in \mathcal{D}(T^+)$

The following are equivalent:

(i) $f \in \operatorname{argmin} \ldots$

(ii) $Tf = P_{\overline{R(T)}}\, g$ — orthogonal projection onto $\overline{R(T)}$

(iii) $T^*Tf = T^*g$ (normal equations)

Proof

(i) $\Rightarrow$ (ii)

$$\|Tf - g\|_Y^2 = \|Tf - P_{\overline{R(T)}}\, g\|_Y^2 + \|g - P_{\overline{R(T)}}\, g\|^2$$

f to minimize $\|Tf-g\|^2 \Leftrightarrow$ make this vanish, which is possible because $g \in \mathcal{D}(T^+)$

(iii) $\Leftrightarrow$ (i) already discussed ∎

(ii) If $g \in \mathcal{D}(T^+)$: $TT^+g = P_{\overline{R(T)}}\, g$

Theorem 1.3.1G (Properties of T^+)

- $R(T^+) = N(T)^\perp$ (simple consequence of minimal norm property)
- $T^+ \in \mathcal{L}(Y, X) \Leftrightarrow R(T) \subset Y$ closed

Proof

"$\Rightarrow$" Lemma 1.3.1F $\Rightarrow$ $TT^+g = P_{\overline{R(T)}}\, g \quad \forall g \in Y$ — possible because $T^+ \in \mathcal{L}(Y,X) \Rightarrow \mathcal{D}(T^+) = Y$

$\Rightarrow \quad R(T) \supset R(TT^+) = R(P_{\overline{R(T)}}) = \overline{R(T)}$

$\Rightarrow$ $R(T)$ must be closed

"$\Leftarrow$" $\tilde{T} = T|_{N(T)^\perp}$ $\quad N(T)^\perp \to R(T)$: bijective linear operator between Hilbert space ($R(T)$ closed $\Rightarrow$ Hilbert space)

$\tilde{T}$ has a bounded inverse by open mapping thm.

$$\|T^+g\|_X = \|\tilde{T}^{-1}\tilde{T}T^+g\|_X \le \|\tilde{T}^{-1}\|\,\|\tilde{T}T^+g\|_Y$$

$g \in \mathcal{D}(T^+) = Y$

(ii) from Lemma 1.3.1F: $\tilde{T}T^+g = P_{\overline{R(T)}}\, g$ =

$= \|\tilde{T}^{-1}\|\,\|P_{\overline{R(T)}}\, g\|_Y \le$

$\le \|\tilde{T}^{-1}\|\,\|g\|_Y \quad \forall g \in Y$

~~densely continuous operator~~

~~$\Rightarrow$ continuous extension of T^+ to Y possible~~

"Definition": Linear operator equation $Tf = g$ is ill-posed, if

T^+ is unbounded

$\Updownarrow$

$R(T) \subset Y$ is not closed

1.3.2 Compact operators

Def 1.3.2A: A linear operator $T: X \to Y$ is compact if for any bounded $U \subset X$
($T(U)$ is relatively compact)
$\overline{T(U)} \subset Y$ is compact

(Equivalent: If $(x_k) \subset X$ is a bounded sequence, the (Tx_k) has a convergent subsequence)

Set of compact operators $X \to Y \cong K(X, Y)$ is a subspace of $\mathcal{L}(X, Y)$.

Theorem 1.3.2B: $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{L}(Y, W)$
[W another Hilbert space]
T or S compact $\Rightarrow ST \in K(X, W)$

(Recall: continuous operators map compact sets to compact sets)

Theorem 1.3.2C $T \in K(X, Y)$ and $\dim R(T) = \infty$
$\Rightarrow R(T)$ not closed!
($\Leftrightarrow Tf = g$ ill-posed)

Proof by contradiction
Assume $R(T)$ is closed, i.e. Hilbert space
$\tilde{T}: N(T)^\perp \to R(T)$ has bounded inverse (open mapping thm.)
$T\tilde{T}^{-1} = Id_{(R(T))}$, but $T\tilde{T}^{-1} \in K(R(T), R(T))$

Known: Id compact in a Banach space if and only if the space has finite dimension $\lightning$ (contradiction)

$\blacksquare$

Theorem 1.3.2D $K(X, Y) \subset \mathcal{L}(X, Y)$ closed (in operator norm)

Application: $(F_n)_n \subset \mathcal{L}(X, Y)$ sequence of operators with finite dimensional range $\Rightarrow F_n \in K(X, Y)$.
$\Rightarrow \lim_{\text{norm}} F_n \in K(X, Y)$, if limit exists

Result: $\mathcal{F}(X,Y) \hat{=}$ bounded finite dimensional operators
$\Rightarrow$ [X, Y separable]: $\overline{\mathcal{F}(X,Y)}^{\mathcal{L}(X,Y)} = K(X,Y)$

(In some sense Y is "too big")

Example: $D \subset \mathbb{R}^d$ bounded,

$$(Tf)(x) = \int_D k(x,y) f(y)\, dy$$

Theorem 13.2E

(i) If $k \in L^2(D \times D)$, ~~$D \subset \mathbb{R}^d$ bounded~~ $D \subset \mathbb{R}^d$
$\Rightarrow T \in K(L^2(D), L^2(D))$

(ii) If $k(x,y) = \hat{k}(x-y)$, $D = \mathbb{R}^d$ with $\hat{k} \in L^1(\mathbb{R}^d)$, supp $\hat{k}$ bounded
$\Rightarrow T \in K(L^2(D), L^2(D))$

Proof

(i) Known from measure theory:
$\exists$ sequence of step functions $(s_n)_n$
with $s_n \to k$ in $L^2(D \times D)$

For each s_n: there is a finite partitioning of $D \times D = \bigcup_j D_j^x \times D_j^y$

$$s_n(x,y) = \sum_{j=1}^{N} s_{n,j}\, \chi_{D_j^x}(x) \chi_{D_j^y}(y)$$

$s_{n,j} \in \mathbb{R}$ characteristic functions

$$(T_n f)(x) = \int_D s_n(x,y) f(y)\, dy = \sum_j s_{n,j} \chi_{D_j^x}(x) \int_D \chi_{D_n^y}(y) f(y)\, dy$$

$\Rightarrow R(T_n) \subset \operatorname{span}\{\chi_{D_1^x}, \dots, \chi_{D_N^x}\} \Rightarrow T_n \in \mathcal{F}(L^2(D), L^2(D))$

$$\|(T - T_n)(f)\|_{L^2}^2 = \int_D \left| \int_D (k - s_n)(x,y) f(y)\, dy \right|^2 dx \overset{CS}{\leq}$$

$$\leq \int_D \left\{ \int_D |(k - s_n)(x,y)|^2 dy \int_D |f(y)|^2 dy \right\} dx \leq$$

$$\overset{\text{Fubini}}{\leq} \|f\|_{L^2(D)}^2 \|k - s_n\|_{L^2(D \times D)}^2$$

$s_n \to k$ in $L^2(D \times D) \Rightarrow T_n \to T$ in $L^2(D)$