1.4 Choice of regularization parameter

$T \in \mathcal{L}(X,Y)$

$\{R_\alpha\}_{\alpha\in A} \subset \mathcal{L}(Y,X)$ induced by filter $\{q_\alpha\}_{\alpha\in A}$

Ill-posed operator equation $Tf = g$, solution $f^+ \in X$, $Tf^+ = g$

[$\Rightarrow g \in R(T)$: exact data attainable]

Recall: if $f^+ \in K_{\nu,\varrho} := \{f \in X : \|f\|_\nu \le \varrho\}$ ("source condition")

$\|f^+ - R_{\alpha^*} g^\delta\|_X \le C_\nu \delta^{\frac{\nu}{\nu+1}} \varrho^{\frac{1}{\nu+1}}$ if $\{q_\alpha\}$ has qualification $\nu_0 \ge \nu$

and $\alpha^* = \alpha^*(\varrho, \delta, \nu) = \left(\frac{C_q \delta}{\nu C_\nu \varrho}\right)^{2/\nu+1}$

[order optimal a priori parameter choice rule

Problem: ϱ not known, α^* is bad if ϱ is underestimated

Desired: P.C.Rs that do not rely on knowledge of ϱ and ν

1.4.1 The discrepancy principle

$\rightarrow$ gives $\bar\alpha = \bar\alpha(\delta, g^\delta)$: a posteriori P.C.R

Def 1.4.1.1 (Morozov discrepancy principle)

Pick $\tau > C_r := \sup\{|r_\alpha(\lambda)| : \alpha \in A,\ 0 \le \lambda \le \|T\|^2\} \ge 1$

(i) If $\underbrace{\frac{\|g^\delta\|_Y}{\delta}}_{\text{signal to noise ratio}} \ge \tau$, pick a safety factor $\eta > 1$ and fix $\alpha_d = \alpha(\delta, g^\delta)$ according to

$$\alpha_d = \sup\{\alpha \in A : \underbrace{\|g^\delta - TR_\alpha g^\delta\|_Y}_{\text{residual: computable!}} \le \tau\delta \le \|g^\delta - TR_{\alpha\eta} g^\delta\|_Y\}$$

(ii) If $\frac{\|g^\delta\|}{\delta} < \tau$ ("useless data")

set "$\alpha_d = \infty$" : $R_{\alpha_d} := 0$

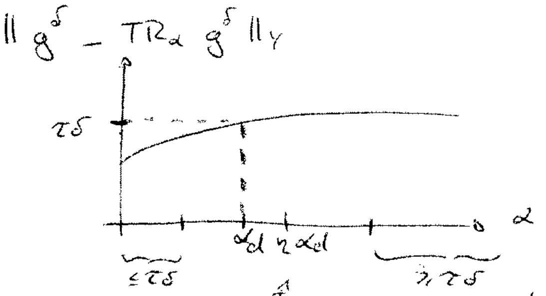
Is α_d well-defined in case (i)?

$$\|g^\delta - TR_\alpha g^\delta\|_Y^2 = \sum_j |r_\alpha(\sigma_j)|^2 \, |(g, g_j)_Y|^2$$

$$\lim_{\alpha\to 0} \| g^\delta - TR_\alpha g^\delta \|_Y^2 = \sum_{j:\sigma_j=0} |(g^\delta, g_j)_Y|^2 = \| P_{\overline{R(T)}^\perp} g^\delta \|_Y^2$$

$$= \| P_{\overline{R(T)}^\perp} (g - g^\delta) \|_Y^2 \le \| g - g^\delta \|_Y^2 \le \delta^2$$

$$\| R_\alpha \| \le \frac{c_q}{\sqrt{\alpha}} \Rightarrow \lim_{\alpha\to\infty} \| g^\delta - TR_\alpha g^\delta \|_Y = \| g^\delta \| \geq \tau\delta$$



we can always find an α_d here

For iterative regularization
$\hat{f}_0 := 0 \to$ not enough
$\vdots$
take first $\hat{f}_k : \| g^\delta - T\hat{f}_k \|_Y \le \tau\delta$

Regularization error / approximation error

$$\| f^+ - R_{\alpha_d} T f^+ \|_X = \sum | r_{\alpha_d}(\sigma_j^2) |^2 \, | (f^+, f_j)_X |^2$$

[Tool: Hölder inequality for sequencies
$p = \nu + 1, \quad q = \frac{\nu+1}{\nu}, \quad \Rightarrow \frac{1}{p} + \frac{1}{q} = 1$]

$$= \sum \left\{ |r_\alpha(\sigma_j^2)|^{2/p} \sigma_j^{-2/q} |(f^+, f_j)_X|^{2/p} \right\} \left\{ |r_\alpha(\sigma_j^2)|^{2/q} \sigma_j^{2/q} |(f^+, f_j)_X|^{2/q} \right\}$$

$$\le \Big(\sum_j \underbrace{|r_\alpha(\sigma_j^2)|^2}_{\le C_r^2} \sigma_j^{-2\nu} |(f^+, f_j)_X|^2 \Big)^{1/p} \Big(\sum_j |r_\alpha(\sigma_j^2)|^2 \sigma_j^2 |(f^+, f_j)_X|^2 \Big)^{1/q}$$

$$\le \left(C_r^2 \| f^+ \|_\nu^2 \right)^{1/p} \left(\| g - TR_\alpha g \|_Y^2 \right)^{1/q}$$

$$\left[\| g - TR_\alpha g \|_Y^2 = \sum_j |r_\alpha(\sigma_j^2)|^2 \, |(\underbrace{g}_{=Tf^+}, g_j)_Y|^2 \overset{T^* g_j = \sigma_j f_j}{=} \sum_j \sigma_j^2 |r(\sigma_j^2)|^2 |(f^+, f_j)_X|^2 \right.$$

$$\| g - TR_{\alpha_d} g \|_Y \le \underbrace{\| (Id - TR_\alpha)(g - g^\delta) \|_Y}_{\le C_r \| g - g^\delta \|_Y} + \underbrace{\| g^\delta - TR_{\alpha_d} g^\delta \|_Y}_{\le \tau\delta}$$

$$\| f^+ - R_{\alpha_d} T f^+ \|_X \le \underbrace{C_r^{1/\nu+1} \, \rho^{1/1+\nu} \left(C_r \delta + \tau\delta \right)^{\nu/\nu+1}}_{C_\rho^{1/1+\nu} \, \delta^{\nu/1+\nu}}$$

Data noise error

$$\| R_{\alpha_d} (g - g^\delta) \|_X \le \frac{c_q}{\sqrt{\alpha_d}} \delta$$

["Show that α is big enough"]

Goal: $\delta / \sqrt{\alpha_d} \le C \rho^{1/1+\nu} \delta^{\nu/1+\nu}$

$\Leftarrow\Rightarrow \quad \alpha \ge c' \delta^{2/\nu+1} \rho^{-2/\nu+1}$ (*)

Known from discrepancy principle: $\| g^\delta - T R_{\eta \alpha_d} g^\delta \| \ge \tau \delta$

$$\Rightarrow \tau \delta \le \| (Id - T R_{\eta \alpha_d}) g \|_Y + \underbrace{\| (Id - T R_{\eta\alpha_d})(g - g^\delta) \|_Y}_{\le c_r \delta}$$

$g = Tf^+$

$$\| (Id - T R_{\eta \alpha_d}) g \|_Y^2 = \sum_j \sigma_j^2 \, | r_{\eta \alpha_d} (\sigma_j^2) |^2 \, | (f^+, f_j)_X |^2$$

$$\le \sum \sigma_j^{2+2\nu} \, | r_{\eta \alpha_d} (\sigma_j^2) |^2$$

$$\le \left(\sup_j \sigma_j^{2+2\nu} | r_{\eta \alpha_d} (\sigma_j^2) |^2 \right) \| f^+ \|_\nu^2$$

Assumption: $\{q_\alpha\}$ has qualification $\ge \nu + 1$

$\Rightarrow \quad \lambda^{\nu+1} | r_\alpha(\lambda) |^2 \le c_\nu^2 \alpha^{1+\nu} \quad \forall \alpha, \; 0 \le \lambda \le \| T \|^2$

$\le c_\nu^2 (\eta \alpha_d)^{1+\nu} \| f^+ \|_\nu^2$

$$\Rightarrow \tau \delta \le c_\nu^2 (\eta \alpha_d)^{\frac{1+\nu}{2}} \rho + c_r \delta$$

$[\tau > c_r]$

$$\Rightarrow (\tau - c_r) \delta \le C \alpha_d^{\frac{1+\nu}{2}} \rho$$

$$\Rightarrow \alpha_d \ge c' \delta^{2/1+\nu} \rho^{-2/\nu+1} \quad \Leftarrow\Rightarrow \quad (*)$$