Stochastic noise: $\Pi_n \left(X_j(\omega)\right)_{j=0}^{n}$ "$=: \varepsilon Z$" (*)

$$\operatorname{Cov}\left(\Pi_n \left(X_j(\omega)\right)_{j=0}^{n}\right) = \Pi_n \underbrace{\operatorname{Cov}\left(\left(X_j(\omega)\right)_{j=0}^{n}\right)}_{\operatorname{Var}(X)\cdot \operatorname{Id}_{\mathbb{R}^{n+1}}} \Pi_n^* = \operatorname{Var}(X)\,\Pi_n \Pi_n^*$$

$$\Pi_n(\underline{y}) = \sum_{j=-n/2}^{n/2} \left(\underline{\underline{F}}^{-1}\underline{y}\right)_j e_j$$

$\underline{y} \in \mathbb{R}^{n+1}$

$\underline{\underline{F}}$: Fourier matrix $\quad (\underline{\underline{F}})_{kl} = e^{\frac{2\pi i}{n+1} kl}$

$$\Pi_n^* : L^2_{per}(0,1) \longrightarrow \mathbb{C}^{n+1}$$

$$(\Pi_n \underline{y}, h)_{L^2} = \sum_j \left(\underline{\underline{F}}^{-1}\underline{y}\right)_j (e_j, h)_{L^2} = \left\langle \left(\underline{\underline{F}}^{-1}\underline{y}\right), \left((e_j,h)_{L^2}\right)_j \right\rangle$$

$$= \left\langle \underline{y}, \underline{\underline{F}}^{-H} (e_j, h)_{L^2} \right\rangle$$

$$\underline{\underline{F}}^{-1}\,\underline{\underline{F}}^{-H} = \frac{1}{n}\operatorname{Id}$$

$$\Rightarrow \Pi_n \Pi_n^* h = \sum_j \frac{1}{n} (e_j, h)_{L^2}\, e_j$$

projection

$$\Rightarrow \operatorname{Cov}\left(\Pi_n \left(X_j(\omega)\right)_{j=0}^{n}\right) = \frac{1}{n} \operatorname{Prj}_{\operatorname{span}\{e_j\}_{j=-n/2}^{n/2}}$$

$$\Rightarrow \| \operatorname{Cov} \ldots \| \le \frac{1}{n}$$

$$\Rightarrow \varepsilon \text{ in } (*) \rightarrow \varepsilon \approx \frac{1}{n}$$

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II. Non-linear inverse problems

2.1 Ill posed non-linear operator equations

X, Y = Hilbert spaces

$\Phi : \mathcal{D}(\Phi) \subset X \longrightarrow Y$

Find $f \in \mathcal{D}(\Phi)$: $\Phi(f) = g$ ($\leftarrow$ data)

Assumption: g is attainable: $g \in \mathcal{R}(\Phi)$

Noisy data: $g^\delta \in Y$

$\|g^\delta - g\|_Y \le \delta$ (noise level)

Def 2.1.B: The non-linear operator equation $\phi(f) = g$ is locally ill-posed at $f^* \in \mathcal{D}(\phi)$, if for any neighborhood $\mathcal{U} \overset{\subset \mathcal{D}(\phi)}{}$ of f^* there is a sequence $(x_n) \in \mathcal{U}$ such that $x_n \not\to f^*$, but $\phi(x_n) \to \phi(f^*)$

($\Leftrightarrow$ no local continuous inverse of ϕ exists at $\phi(f^*)$)

Lemma 2.1.C $T \in \mathcal{L}(X, Y)$ is locally ill-posed, if and only if $R(T)$ not closed or $N(T) \neq \{0\}$

Proof: $\Leftarrow$: if $N(T) \neq \{0\}$: $x_n = f^* + \rho_n v$, $v \in N(T) \setminus \{0\}$ with $\rho_n \to \rho \neq 0$ for $n \to \infty$

$T(x_n) = T(f^*)$ $\forall n$

- If $R(T) \neq \overline{R(T)}$ $\overset{\text{Sect 1.2}}{\Rightarrow}$ $T^+ : D(T^+) = R(T) + R(T)^\perp \to X$ is not bdd.

There is $(y_n)_n \in R(T)$: $\|y_n\|_Y = 1$, $\|T^+ y_n\|_X \to \infty$

$$x_n = f^* + \frac{T^+ y_n}{\|T^+ y_n\|_X} \not\to f^*, \text{ but } Tx_n = Tf^* + \frac{y_n}{\|T^+ y_n\|_X} \to Tf^*$$

$\Rightarrow$ clear

Local ill-posedness and linearization: $\overset{\circ}{\mathcal{D}}(\phi)$ = interior of $\mathcal{D}(\phi)$

$D\phi(f) \mathrel{\hat{=}}$ Fréchet derivative of ϕ at $f \in \overset{\circ}{\mathcal{D}}(\phi)$. $D\phi(f) \in \mathcal{L}(X, Y)$

$\|\phi(f+h) - \phi(f) - D\phi(f)(h)\| = o(\|h\|)$, $h \in X$

Mean value formula: $\phi(f+h) - \phi(f) = \int_0^1 D\phi(f + \tau h) h \, d\tau$

Thm 2.1.D: $\phi : \mathcal{D}(\phi) \subset X \to Y$ is differentiable, $D\phi : \mathcal{D}(\phi) \to \mathcal{L}(X, Y)$ is locally Lipschitz-continuous, $\phi(f) = g$ is locally ill-posed at $f^* \in \overset{\circ}{\mathcal{D}}(\phi)$. Then $R(D\phi(f^*))$ is not closed <u>or</u> $N(D\phi(f^*)) \neq \{0\}$

Proof (indirect) assume $R(A)$ closed and $N(A) = \{0\}$ where $A = D\phi(f^*)$

$A^{-1} : R(A) \to X$ bounded $\Rightarrow$ $\underbrace{\exists \gamma > 0 : \|Ax\|_Y \geq \gamma \|x\|_X \ \forall x \in X}_{(*)}$

Mean value formula (f close to f^*) $\Rightarrow$ $\|\phi(f) - \phi(f^*) - D\phi(f^*)(f - f^*)\| =$

$= \|\int_0^1 [D\phi(f^* + (f - f^*)\tau) - D\phi(f^*)](f - f^*) d\tau$

By local Lipschitz property, $L > 0$ $\leq \int_0^1 L \|(f - f^*)\tau\|_X \|f - f^*\|_X \, d\tau$

$= 1/2 \; L \; \|f - f^*\|_X^2$

$\overset{\Delta\text{-ineq}}{\Rightarrow} \; \|A(f-f^*)\|_Y - \|\Phi(f) - \Phi(f^*)\|_Y \le \frac{1}{2} L \|f - f^*\|_X^2$

$\circledast \Rightarrow \; \gamma \|f - f^*\|_X - \frac{1}{2} L \|f - f^*\|_X^2 \le \|\Phi(f) - \Phi(f^*)\|_Y$

$[\, \|f - f^*\|_X \le \frac{1}{L}\gamma \;\Rightarrow$ defines $\mathcal{U}$

$\Rightarrow \frac{1}{2}\gamma \|f - f^*\|_X \le \|\Phi(f) - \Phi(f^*)\|_Y$, which contradicts local ill-posedness $\square$

Bad news: If $R(D\Phi(f^*))$ not closed $\not\Rightarrow$ $\Phi(f) = g$ is locally ill-posed at f^*

Example: distributed parameter estimation

on $]0,1[$: $-u'' + f(x)u = s \in L^2(0,1)$, $u(0) = u(1) = 0$

Data u (temperature), sought f

Setting: $X = Y = L^2(0,1)$

$\Phi : \mathcal{D}(\Phi) \subset X \to Y$ defined by $\Phi(f) := u$

$\mathcal{D}(\Phi) := \{ f \in L^2(0,1) : 0 \le f(x) \le f_{max} \text{ a.e.} \}$

Noisy data $g^\delta \mapsto u^\delta$: $\|u - u^\delta\|_{L^2(0,1)} \le \delta$

Formally: $f = \frac{s + u''}{u}$ if $u \ne 0$

ill-posed in L^2-setting

Def 2.1.E $\Phi : \mathcal{D}(\Phi) \subset X \to Y$ is compact if the image of any bounded sequence in $\mathcal{D}(\Phi)$ contains a convergent subsequence

Def 2.1.F $\Phi : \mathcal{D}(\Phi) \subset X \to Y$ is weakly closed if for all sequences $(x_n) \subset \mathcal{D}(\Phi)$

$$\left.\begin{array}{l} x_n \rightharpoonup x \\ \Phi(x_n) \rightharpoonup y \end{array}\right\} \Rightarrow \begin{array}{l} x \in \mathcal{D}(\Phi) \\ y = \Phi(x) \end{array}$$

Thm 2.1.G if $\dim X = \infty$, Φ continuous, compact and weakly closed $\Rightarrow$ $\Phi(f) = g$ is locally ill-posed at all $f \in \mathring{\mathcal{D}}(\Phi)$

probably not necessary

Proof: $\{e_n\}$ ONB: $\forall f \in X \quad ((f, e_n)_X)_n \to 0$

$\Rightarrow e_n \rightharpoonup 0$

$f \in \mathring{\mathcal{D}}(\Phi)$: ϱ-neighborhood of $f \subset \mathcal{D}(\Phi)$: $x_n = f + \varrho e_n$

$x_n \rightharpoonup f$ but $x_n \not\to f$

Assume $\Phi(x_n) \not\to \Phi(f)$

$\exists$ subsequence and $\varepsilon > 0$ $\quad \| \Phi(x_{n_k}) - \Phi(f) \|_Y \geq \varepsilon$

weakly cvg $\Rightarrow$ bdd (clear)

Φ cpt $\Rightarrow \Phi(x_{n_k})$ contains convergent subsequence

(abuse of notation)

$\Phi(x_{n_k}) \to \tilde{y} \in Y$ and $\| \tilde{y} - \Phi(f) \| \geq \varepsilon$

$x_{n_k} \rightharpoonup f$ ↯ to weak closedness □

For distributed parameter estimation:

$$\int_0^1 u'v' + f(x)\, uv \, dx = \int_0^1 sv \, dx \quad \forall v \in H_0^1(0,1)$$

$[0 \leq f \leq f_{max}$ a.e.$]$ $\qquad \| u \|_{H^1(0,1)} \leq C \| s \|_{L^2(0,1)}$

$$\| u'' \|_{L^2} \leq f_{max} \| u \|_{L^2} + \| s \|_{L^2}$$

Compactness of Φ: $\Phi(f) = u$

$(f_n)_n \subset \mathcal{D}(\Phi)$ bdd $\Rightarrow \| \Phi(f_n) \|_{H^2(0,1)}$ bdd

$H^2(0,1) \overset{c}{\hookrightarrow} L^2(0,1)$ [Rellich]

$(\Phi(f_n))_n$ contains L^2-convergent subsequence

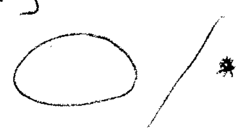
Φ weakly closed:

- $[\mathcal{D}(\Phi)$ is weakly sequentially closed$]$
  - $\mathcal{D}(\Phi) \subset L^2(0,1)$ closed, because
  - [Thm: closed, convex subsets of a Hilbert space are weakly closed $\to$ Hahn-Banach thm]

if a sequence converges in L^2, it also converges pointwise a.e.

see measure theory

separating hyperplane



$f_n \to \tilde{f}$ in $L^2(0,1)$

$\Phi(f_n) \to \tilde{u}$ in $L^2(0,1)$