<u>Proof:</u> for all $f \in \mathcal{D}(\Phi)$ (A)

$$\|\Phi(f_n) - g_n\|_Y^2 + \alpha \|f_n - f^*\|_X^2 = J_\alpha(f_n, g_n) \le J_\alpha(f, g_n)$$

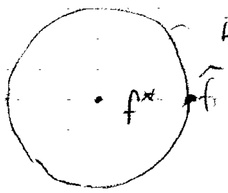
$\to (f_n) \subset X$, $(\Phi(f_n)) \subset Y$ bounded

$$\Rightarrow \begin{cases} f_n \to \hat{f} \in \mathcal{D}(\Phi) \\ \Phi(f_n) \to \hat{g} = \Phi(\hat{f}) \end{cases} \quad \text{[sub-sequencies !]}$$

As before: $J_\alpha(\hat{f}, g^\delta) \le \liminf\limits_{n\to\infty} J_\alpha(f_n, g_n)$

$$\underset{\text{minimizer}}{\uparrow} \qquad \overset{(A)}{\le} \lim_{n\to\infty} J_\alpha(f, g_n) = J_\alpha(f, g^\delta)$$

To show $f_n \to \hat{f}$ in X



$B_{\|f^* - f^\delta\|}(f^*)$ convex

In Hilbert space X, $C \subset X$ convex, closed, $(x_n) \subset C$, $x_n \rightharpoonup x \in C$

$\|x\|_X = \inf\limits_{y \in C} \|y\|_X \Rightarrow x_n \to x$

⌈Proof based on $\underset{\substack{x \ne y \\ \|x\| = \|y\| = 1}}{x, y \in H} \Rightarrow \|1/2\,(x+y)\| < 1$

⌊

~~Assume $f_n \not\to \hat{f} \Rightarrow \exists$ subsequence~~ with $\|f_n - f^*\|_X > \|\hat{f} - f^*\|_X$ ~~subsequence~~

~~$\lim\limits_{n\to\infty} \|\Phi(f_n) - g_n\|_Y^2 \le$~~

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So far we could find subsequence (f_n):

$$f_n \rightharpoonup \hat{f} \in \mathcal{D}(\Phi)$$
$$\Phi(f_n) \rightharpoonup \Phi(\hat{f})$$
$$\lim_{n\to\infty} J_\alpha(f_n, g_n) = J_\alpha(\hat{f}, g^\delta) \qquad (*)$$

Assume $f_n \not\to \hat{f}$ in X

$\Rightarrow$ (up to selection of subsequence)

$\|f_n - f^*\|_X \ge \|\hat{f} - f^*\|_X + \varepsilon$ for some $\varepsilon > 0$

$$(*) \quad \lim_{n\to\infty} \left\{ \|\Phi(f_n) - g_n\|_Y^2 + \alpha \|f_n - f^*\|_X^2 \right\} = \|\Phi(\hat{f}) - g^\delta\| + \alpha \|\hat{f} - f^*\|_X^2$$

$$\Rightarrow \lim_{n\to\infty} \|\Phi(f_n) - g_n\|_Y^2 \le \|\Phi(\hat{f}) - g^\delta\|_Y^2 + \alpha \underbrace{\left(\|\hat{f} - f^*\|_X^2 - \lim_{n\to\infty} \|f_n - f^*\|\right)}_{\le -\varepsilon}$$

$$\le \|\Phi(\hat{f}) - g^\delta\|_Y^2 - \alpha\varepsilon$$

$$\Leftrightarrow \|\Phi(\hat{f}) - g^\delta\|_Y = \liminf_{n\to\infty} \|\Phi(f_n) - g_n\|_Y \ \to \ ↯$$

If $J_\alpha(f, g^\delta)$ has unique minimizer
$\Rightarrow \hat{f}$ unique and all sequences of minimizer converge to it strong

<u>Thm (2.2.E)</u> Let $\bar{\alpha}: \mathbb{R}^+ \to \mathbb{R}^+$ be an a-priori parameter choice rule with
$$\lim_{\delta \to 0} \bar{\alpha}(\delta) = 0 \quad , \lim_{\delta \to 0} \delta^2 / \bar{\alpha}(\delta) = 0$$

Then for every sequence $(g^\delta)_{\delta > 0}$, $\| g^\delta - g \| < \delta$
$(R_{\bar{\alpha}(\delta)}(g^\delta))_{\delta > 0}$ has a convergent subsequence

- The limit of every convergent subsequence is an f^* - output least squares solution of $\Phi(f) = g$
- If the f^* - OLS is unique, then
$$\lim_{\delta \to 0} R_{\bar{\alpha}(\delta)}(g^\delta) = f^+$$

<u>Proof:</u> $f_\delta := R_{\bar{\alpha}(\delta)} g^\delta \in \mathcal{D}(\Phi) \;\hat{=}$ minimizer $J_{\bar{\alpha}(\delta)}(f, g^\delta)$

$$J_\alpha(f_\delta, g^\delta) = \| \Phi(f_\delta) - g^\delta \|_{Y^+}^2 + \bar{\alpha}(\delta) \| f_\delta - f^* \|_X^2 \leq J_{\bar{\alpha}(\delta)}(f^+, g^\delta) \leq \delta^2 + \bar{\alpha}(\delta) \| f^+ - f^* \|_X^2$$

(i) $\delta \to 0 \Rightarrow \lim_{\delta \to 0} \| \Phi(f_\delta) - g^\delta \|_Y = 0 \Rightarrow \lim_{\delta \to 0} \Phi(f_\delta) = g^\delta$

(ii) $\| f_\delta - f^* \|_X^2 \leq \underbrace{\frac{\delta^2}{\bar{\alpha}(\delta)}}_{\to 0 \text{ for } \delta \to 0} + \| f^+ - f^* \|_X^2 \overset{(A)}{\Rightarrow} \underbrace{(f_\delta)_\delta \text{ bdd}}$

$\exists$ subsequence (f_k)
$f_k \rightharpoonup \tilde{f} \in \mathcal{D}(\Phi)$
$\Phi(f_k) \rightharpoonup g$

$$\| \tilde{f} - f^* \|_X \leq \liminf_{k \to \infty} \| f_k - f^* \|_X \overset{(A)}{\leq} \| f^+ - f^* \|_X$$

$\Rightarrow \tilde{f}$ is f^*-output least squares solution

Skipped: $f_k \to \tilde{f}$ strongly

□

<u>Rate of convergence</u>

- Φ is Fréchet-differentiable in a neighborhood of f^+ with Lipschitz continuous derivative. $\exists L > 0$.
$$\| D\Phi(f) - D\Phi(f^+) \|_{\mathcal{L}(X,Y)} \leq L \| f - f^+ \|_X$$
$$\forall f \in \mathring{\mathcal{D}}(\Phi), \| f - f^+ \| < r \quad \text{with} \quad r > 2 \| f^+ - f^* \|_X$$

- $D\phi(f^+)$ is compact with singular system $(\sigma_j, f_j, g_j)_j$
- Hölder-type source condition (a priori knowledge)
  $f^+ - f^* \in K_{1,\varrho} := \{ f : \sum_j \sigma_j^{-2} |(f, f_j)|^2 \le \varrho^2 \}$
- Smallness assumption: $L \cdot \varrho < 1$
- Parameter choice rule: $\bar{\alpha}(\delta) = \delta$

$\boxed{1}$ $f_\delta := R_{\bar\alpha(\delta)}(g^\delta) \Rightarrow J_{\bar\alpha}(f_\delta, g^\delta) \le J_{\bar\alpha}(f^+, g^\delta) = \delta^2 + \bar\alpha \|f^+ - f^*\|_X^2$

↑ sequence converging to g

$$\|\phi(f_\delta) - g^\delta\|_Y^2 + \bar\alpha \|f_\delta - f^*\|_X^2 + \bar\alpha \|f_\delta - f^+\|_X^2 \le \delta^2 + \bar\alpha \|f^+ - f^*\|_X^2 + \bar\alpha \|f_\delta - f^+\|_X^2$$

$$\Rightarrow \boxed{\|\phi(f_\delta) - g^\delta\|_Y^2 + \bar\alpha \|f_\delta - f^+\|_X^2} \le \delta^2 + \bar\alpha \|f^+ - f^*\|_X^2 + \bar\alpha \|f_\delta - f^+\|_X^2 - \bar\alpha \|(f_\delta - f^+) + (f^+ - f^*)\|_X^2$$

$$= \boxed{\delta^2 + \bar\alpha\, 2 (f_\delta - f^+, f^+ - f^*)_X} \quad (*)$$

$$\le \delta^2 + 2\bar\alpha \|f_\delta - f^+\|_X \|f^+ - f^*\|_X$$

Estimate of the type
$\gamma^2 \le \delta^2 + \gamma c$, $c > 0$
$\Rightarrow \gamma \le \delta + c$

$$\Rightarrow \|f_\delta - f^+\|_X^2 \le \frac{\delta}{\sqrt{\bar\alpha}} + 2\|f^+ - f^*\|_X$$

For small $\delta \Rightarrow \|f_\delta - f^+\|_X \le r$

$\boxed{2}$ Rewriting source condition:

$$\sum_j \sigma_j^{-2} |(f_j, f^+ - f^*)_X|^2 \le \varrho^2$$

$$w := \sum_j \sigma_j^{-1} (f_j, f^+ - f^*)_X\, g_j \in Y$$

↑ square summable

$\|w\|_Y \le \varrho$ $\quad (**)$

$$\left[D\phi(f^+)^* w = \sum_j \sigma_j^{-1} (f_j, f^+ - f^*) \underbrace{D\phi(f^+)^* g_j}_{= \sigma_j f_j} = f^+ - f^* \right]$$

$\boxed{3}$ $(*)$ & $(**)$ $\Rightarrow \|\phi(f_\delta) - g^\delta\|_Y^2 + \alpha \|f_\delta - f^*\|_X^2 \le \delta^2 + \bar\alpha 2 (f_\delta - f^+, D\phi(f^+)^* w)$

$$= \delta + \bar\alpha\, 2 (D\phi f(f^+)(f_\delta - f^+), w)_Y$$

Idea: introduce local linearization

$$= \delta - 2\bar\alpha (\phi(f_\delta) - \phi(f^+) - D\phi(f^+)(f_\delta - f^+), w)_Y + 2\bar\alpha (\phi(f_\delta) - \phi(f^+), w)$$

Idea: Mean value thm

$$\| \Phi(f_\delta) - \Phi(f^+) - D\Phi(f^+)(f_\delta - f^+)\|_Y =$$

$$= \left\| \int_0^1 \left(D\Phi\left(f^+ + \tau(f^\delta - f^+)\right) - D\Phi(f^+)\right)(f_\delta - f^+)\, d\tau \right\|_Y$$

$$\overset{\text{Lip. cont.}}{\le} \int_0^1 L\tau \|f_\delta - f^+\|_X^2 \, d\tau = \frac{L}{2} \|f_\delta - f^+\|_X^2$$

$$\Rightarrow \| \Phi(f_\delta) - g^\delta \|_Y^2 + \bar\alpha \|f_\delta - f^+\|_X^2 \le \delta^2 + 2\bar\alpha \frac{L}{2} \|f_\delta - f^+\|_X^2 \|w\|_Y + 2\bar\alpha \|\Phi(f_\delta) - g\|_Y \|w\|_Y$$

$$[\, L \|w\|_Y < 1 \,]$$

$$\Rightarrow \| \Phi(f_\delta) - g^\delta \|_Y^2 + \bar\alpha \underbrace{(1 - L\rho)}_{>0} \|f_\delta - f^+\|_X^2 \le \delta^2 + 2\bar\alpha \|\Phi(f_\delta) - \underbrace{g}_{\pm g^\delta}\|_Y \rho$$

$$+ \bar\alpha^2 \rho^2 \qquad\qquad + \bar\alpha^2 \rho^2$$

$$\Rightarrow \left(\|\Phi(f_\delta) - g\|_Y - \alpha\rho \right)^2 + \alpha(1 - L\rho) \|f_\delta - f^+\|_X^2 \le \delta^2 + \bar\alpha^2 \rho^2 + \bar\alpha 2 \delta\rho$$

$$\bar\alpha = \delta \Rightarrow \|f_\delta - f^+\|_X^2 \le \frac{1}{(1 - L\rho)} \left(\delta + \delta\rho^2 + \delta\rho \right) = O(\delta)$$

$$\Rightarrow \text{Rate} \quad O\left(\delta^{1/2}\right)$$

same rate as linear Tikhonov!