ETH Zürich HS 2013
D-MATH
Prof. J. Teichmann

Mathematical Finance

Exercise Sheet 1

Exercise 1-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,\dots,T}, \mathbb{P})$ be a filtered probability space, and $\bar{S} = (S, 1) = (S_k^1, \dots, S_k^d, 1)_{k=0,\dots,T}$ a discrete-time model with time horizon $T \in \mathbb{N}$, i.e., S is adapted to $(\mathscr{F}_k)_{k=0,\dots,T}$. A stochastic process $\varphi := (\vartheta, \eta) = (\vartheta_k^1, \dots, \vartheta_k^d, \eta_k)_{k=0,\dots,T}$ is called a *strategy* if ϑ is predictable for $(\mathscr{F}_k)_{k=0,\dots,T}$ with $\vartheta_0 = 0$, and η is adapted to $(\mathscr{F}_k)_{k=0,\dots,T}$. It is called *affordable (for $\bar{S}$)* if

$$(\varphi_{k+1} - \varphi_k)^{tr} \bar{S}_k = (\vartheta_{k+1} - \vartheta_k)^{tr} S_k + (\eta_{k+1} - \eta_k) \leq 0, \quad k = 0, \dots, T-1.$$

(a) Give an economic interpretation of an affordable strategy.

(b) Show that a strategy $\varphi = (\vartheta, \eta)$ is affordable if and only if there exists an adapted, increasing process $K = (K_k)_{k=0,\dots,T}$ null at 0 such that

$$V_k(\varphi) := \varphi_k \cdot \bar{S}_k = V_0(\varphi) + \sum_{j=1}^{k} \vartheta_j \Delta S_j - K_k, \quad k = 0, \dots, T.$$

Moreover show that K is predictable if and only if η is so.

(c) Show that for all triplets (V_0, ϑ, K), where V_0 is $\mathscr{F}_0$-measurable, $\vartheta = (\vartheta_k^1, \dots, \vartheta_k^d)_{k=0,\dots,T}$ is predictable with $\vartheta_0 = 0$ and $K = (K_k)_{k=0,\dots,T}$ is an adapted, increasing process null at 0, there exists a unique affordable strategy $\varphi = (\vartheta, \eta)$ such that $\varphi_0 \cdot \bar{S}_0 = \eta_0 = V_0$ and $V_k(\varphi) = V_0(\varphi) + \sum_{j=1}^{k} \vartheta_j \Delta S_j - K_k$ for $k = 1, \dots, T$.

Exercise 1-2

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $(Y_k)_{k=1,\dots,T}$ with $T \in \mathbb{N}$ a sequence of independent, non-trivial and strictly positive random variables. Define a discrete-time model $\bar{S} = (S_k, 1)_{k=0,\dots,T}$ with time horizon T by $S_k = \prod_{j=1}^{k} Y_j$, $k = 0, \dots, T$, and let the filtration $(\mathscr{F}_k)_{k=0,\dots,T}$ be generated by S.

(a) Fix $k \in \{1, \dots, T\}$, and assume that $\operatorname{ess\,inf} Y_k < 1 < \operatorname{ess\,sup} Y_k$. Show that there exists a probability measure $\mathbb{Q}^k \approx \mathbb{P}$ on $\mathscr{F}_T$ with $\mathbb{E}_{\mathbb{Q}^k}[Y_k] = 1$ and $\frac{d\mathbb{Q}^k}{d\mathbb{P}} = g_k(Y_k)$ for some measurable function $g_k : (0, \infty) \to (0, \infty)$.

Hint: Show more generally that if Y is a real-valued not-trivial random variable and $\mu \in (\operatorname{ess\,inf} Y, \operatorname{ess\,sup} Y)$, then there exist $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}$ such that $Y \in L^1(\mathbb{Q})$ and $\mathbb{E}_{\mathbb{Q}}[Y] = \mu$ and a measurable function $g : \mathbb{R} \to (0, \infty)$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} = g(Y)$.

Divide the proof into two steps: First, show that there exists $\widetilde{\mathbb{P}} \approx \mathbb{P}$ on $\mathscr{F}$ such that $Y \in L^1(\widetilde{\mathbb{P}})$. Then, set $A := \{Y \leq \mu\}$, use the two conditional probabilities $\widetilde{\mathbb{P}}[\,\cdot \mid A]$ and $\widetilde{\mathbb{P}}[\,\cdot \mid A^c]$ to construct $\mathbb{Q} \approx \widetilde{\mathbb{P}}$ with $Y \in L^1(\mathbb{Q})$ and $\mathbb{E}_{\mathbb{Q}}[Y] = \mu$, and put everything together.

(b) Show that S satisfies NA if and only if $\operatorname{ess\,inf} Y_k < 1 < \operatorname{ess\,sup} Y_k$ for all $k = 1, \dots, T$.

Hint: For "$\Leftarrow$" construct an equivalent martingale measure using part (a). For "$\Rightarrow$", argue by contraposition and construct an explicit arbitrage strategy.

Exercise 1-3

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, and let $B = (B_t^1, B_t^2, B_t^3)_{t\in[0,1]}$ be a three-dimensional Brownian motion starting at $(1, 0, 0)$. Define the process $X = (X_t)_{t\in[0,1]}$ by $X_t := \|B_t\|$, $t \in [0, 1]$, where $\|\cdot\|$ denotes the Euclidean norm.

(a) Show that $\inf_{t\in[0,1]} X_t > 0$ and that X satisfies the SDE

$$\mathrm{d}X_t = \frac{1}{X_t}\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 1, \tag{$*$}$$

where $W = (W_t)_{t\in[0,1]}$ is a one-dimensional Brownian motion starting at 0.

Hint: Use the fact that $\mathbb{P}\big[B_t = a \text{ for some } t \in [0, 1]\big] = 0$ for all $a \neq (1, 0, 0)$ and Lévy's characterisation of Brownian motion.

Let $\bar{S} = (S, 1) = (S_t^1, 1)_{t\in[0,1]}$ be a continuous-time model with time horizon $T = 1$, where S satisfies the SDE

$$\mathrm{d}S_t = S_t\left(\left(\frac{1}{X_t} + 2\right)\mathrm{d}t + \mathrm{d}W_t\right), \quad S_0 = s_0 > 0,$$

and where W is the Brownian motion from $(*)$.

(b) Show that S fails NA by showing that the strategy $\vartheta = (\vartheta_t)_{t\in[0,1]}$, given by $\vartheta_t := \frac{1}{S_t}$, $t \in [0, 1]$, is an arbitrage opportunity.

Exercise 1-4

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $X := (X_t)_{t\in[0,T]}$ be a a compound Poisson process with jump intensity $\lambda > 0$ and jump distribution ν, i.e., $X_t := \sum_{k=1}^{N_t} Y_k$, where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with rate λ and $(Y_k)_{k\in\mathbb{N}}$ a sequence of random variables independent of N such that the Y_k are i.i.d. with distribution ν (with $\nu(\{0\}) = 0$). Suppose about the filtration $(\mathscr{F}_t)_{t\geq 0}$ that X is a Lévy process with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. Let $\widetilde{\lambda} > 0$ and let $\widetilde{\nu} \approx \nu$ be an equivalent probability measure on $\mathbb{R}$. Define the exponential Lévy process $Z = (Z_t)_{t\in[0,T]}$ by

$$Z_t := \exp\left(\sum_{k=1}^{N_t} \varphi(Y_k) + (\lambda - \widetilde{\lambda})t\right),$$

where $\varphi(x) = \log\left(\frac{\widetilde{\lambda}}{\lambda}\frac{\mathrm{d}\widetilde{\nu}}{\mathrm{d}\nu}(x)\right)$.

(a) Show that Z is a $\mathbb{P}$-martingale.

(b) Define the probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$ by $\mathrm{d}\mathbb{Q} = Z_T\,\mathrm{d}\mathbb{P}$. Show that under $\mathbb{Q}$, X is again a compound Poisson process for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ with rate $\widetilde{\lambda}$ and jump distribution $\widetilde{\nu}$.

Hint: Show that X is a Lévy process under $\mathbb{Q}$ for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$, and calculate the characteristic function of X_1 under $\mathbb{Q}$ to determine its law (assuming without loss of generality that $T \geq 1$).

Exercise 1-5

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $R = (R_t)_{t\in[0,T]}$ be a *jump-diffusion*, i.e., there exist a Brownian motion $W = (W_t)_{t\in[0,T]}$ and an independent compound Poisson process $X = (X_t)_{t\geq 0}$ with jump intensity $\lambda > 0$ and jump distribution ν (with $\nu(\{0\}) = 0$) such that $R_t = at + \sigma W_t + X_t$, $t \in [0,T]$, where $a \in \mathbb{R}$ and $\sigma \geq 0$. Suppose about the filtration that R is a Lévy process with respect to $(\mathscr{F}_t)_{t\in[0,T]}$, and suppose about ν that $\nu((-\infty,-1]) = 0$, i.e. the jumps of R are strictly greater than -1. Define the process $S = (S_t)_{t\in[0,T]}$ by $\mathrm{d}S_t := S_{t-}\,\mathrm{d}R_t$, $S_0 = s_0 > 0$, i.e., $S = s_0\mathcal{E}(R)$.

(a) Suppose that R is a martingale for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Show that S is then also a martingale (and not only a local martingale) for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$.

Hint: Use the formula from Exercise 0-3 for $\mathcal{E}(R)$.

(b) Show that in general S satisfies NA if and only if the paths of R are not monotone.

Hint: To establish "$\Leftarrow$", use Exercise 1-4 (b) to show first that there exists a measure $\widetilde{\mathbb{P}} \approx \mathbb{P}$ on $\mathscr{F}_T$ such that the process R is $\widetilde{\mathbb{P}}$-integrable. Then distinguish the cases $\sigma > 0$ and $\sigma = 0$ and apply a second change of measure. Moreover, use that if A and B are independent stochastic processes under $\mathbb{P}$ and $\mathbb{Q} \approx \mathbb{P}$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \Phi(B)$ for some measurable functional Φ, then A and B are still independent under $\mathbb{Q}$ and $\mathcal{L}(A \mid \mathbb{Q}) = \mathcal{L}(A \mid \mathbb{P})$.