ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 2

Exercise 2-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and $\bar{S} = (1, S_t^1, \dots, S_t^d)_{t\in[0,T]}$ a general continuous-time financial market with time horizon $T > 0$.

(a) Show that if there exists an arbitrage opportunity

$$\vartheta = \sum_{k=1}^{N} h^k \mathbb{1}_{]\!]\tau_{k-1},\tau_k]\!]} \in \mathbf{b}\mathcal{E},$$

then there also exists a "one-step buy-and-hold" arbitrage opportunity $\vartheta^* = h\mathbb{1}_{]\!]\sigma_0,\sigma_1]\!]} \in \mathbf{b}\mathcal{E}$.

(b) Assume that S is a semimartingale and satisfies NA. Prove that if $\vartheta \in \Theta_{\text{adm}}$ satisfies $G_T(\vartheta) \geq -c$ $\mathbb{P}$-a.s. for some $c \geq 0$, then $G_\cdot(\vartheta) \geq -c$ $\mathbb{P}$-a.s.

Exercise 2-2

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, $S = (S^1, \dots, S_t^d)_{t\in[0,T]}$ a d-dimensional semimartingale and $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$ an equivalent probability measure.

(a) Assume that $\mathbb{Q}$ is a separating measure for S. Show that if S is (locally) bounded, then $\mathbb{Q}$ is an equivalent (local) martingale measure for S.

Hint: Consider suitable simple integrands.

(b) Assume that $\mathbb{Q}$ is an equivalent σ-martingale measure for S. Show that it is also an equivalent separating measure.

Hint: Use the Ansel-Stricker theorem.

(c) Now assume that $d = 1$, that $(\mathscr{F}_t)_{t\in[0,T]}$ is the natural (completed) filtration of S and that $S = (S_t)_{t\in[0,T]}$ is of the form

$$S_t = \begin{cases} 0 & \text{for } 0 \leq t < T, \\ X & \text{for } t = T, \end{cases}$$

where X is normally distributed with mean $\mu \neq 0$ and variance $\sigma^2 > 0$. Show that in this case the class $\mathcal{M}_{\text{sep}}$ of equivalent separating measures for S is strictly bigger than the class $\mathcal{M}_\sigma$ of equivalent σ-martingale measures for S.

Hint: Describe explicitly all adapted and predictable processes for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$.

Exercise 2-3

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,1}, \mathbb{P})$ be a filtered probability space and $\bar{S} = (1, S_k^1, \dots, S_k^d)_{k=0,1}$ a one-period model. Assume that $\mathscr{F}_0 = \{\emptyset, \Omega\}$.

(a) Show that the set

$$\mathcal{Q} := \left\{ \mathbb{Q} \approx \mathbb{P} \text{ on } \mathscr{F}_1 : \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \text{ is bounded and } \mathbb{E}_{\mathbb{Q}}[|S_1^i|] < \infty \text{ for all } i = 1, \dots, d \right\}$$

is nonempty and convex.

(b) Define the set

$$\mathcal{C} := \{\mathbb{E}_{\mathbb{Q}}[\Delta S_1] : \mathbb{Q} \in \mathcal{Q}\},$$

where $\mathbb{E}_{\mathbb{Q}}[\Delta S_1]$ is a shorthand notation for the vector $\left(\mathbb{E}_{\mathbb{Q}}[\Delta S_1^1], \dots, \mathbb{E}_{\mathbb{Q}}[\Delta S_1^d]\right)^{tr}$. Show that if there exists $\vartheta \in \mathbb{R}^d$ with $\vartheta^{tr} x \geq 0$ for all $x \in \mathcal{C}$, then $\vartheta^{tr} \Delta S_1 \geq 0$ $\mathbb{P}$-a.s.

Hint: Fix $\mathbb{Q} \in \mathcal{Q}$, and for $\epsilon \in (0,1)$, define the function

$$\varphi_\epsilon = \epsilon \mathbb{1}_{\{\vartheta^{tr} \Delta S_1 \geq 0\}} + (1 - \epsilon) \mathbb{1}_{\{\vartheta^{tr} \Delta S_1 < 0\}}$$

and consider the limit $\lim_{\epsilon \to 0} \mathbb{E}_{\mathbb{Q}}[\varphi_\epsilon \vartheta^{tr} \Delta S_1]$.

(c) Show that S satisfies NA if and only if there exists $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ bounded and such that S is a $\mathbb{Q}$-martingale, i.e., S^i is a $\mathbb{Q}$-martingale for each $i = 1, \dots, d$.

Hint: For "$\Rightarrow$" use without proof that if $\mathcal{C}$ is a nonempty convex subset of $\mathbb{R}^d$ with $0 \notin \mathcal{C}$, then there exist $\vartheta \in \mathbb{R}^d$ and $x_0 \in \mathcal{C}$ such that $\vartheta^{tr} x \geq 0$ for all $x \in \mathcal{C}$ and $\vartheta^{tr} x_0 > 0$.

Exercise 2-4

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space. For an increasing, adapted, continuous process $A = (A_t)_{t \in [0,T]}$ null at 0, define the σ-finite measure $\mathbb{P} \otimes A$ on $(\bar{\Omega}, \mathcal{P})$, where $\bar{\Omega} := \Omega \times (0, T]$ and $\mathcal{P}$ denotes the predictable σ-field, by

$$(\mathbb{P} \otimes A)[\bar{N}] := \mathbb{E}\left[\int_0^T \mathbb{1}_{\bar{N}}(\omega, s) \,\mathrm{d}A_s\right], \quad \bar{N} \in \mathcal{P}.$$

(a) Let A^1 and A^2 be increasing, adapted, continuous processes null at 0 with $\mathbb{P} \otimes A^1 = \mathbb{P} \otimes A^2$ on $(\bar{\Omega}, \mathcal{P})$. Show that $A^1 = A^2$.

Hint: Show that $A^1 - A^2$ is a local martingale.

(b) Let B and C be increasing, adapted, continuous processes null at 0. Show that there exist a predictable process $H \in L(C)$ and $\bar{N} \in \mathcal{P}$ such that

$$B_t = \int_0^t H_s \,\mathrm{d}C_s + \int_0^t \mathbb{1}_{\bar{N}} \,\mathrm{d}B_s, \quad \text{and} \quad \int_0^t \mathbb{1}_{\bar{N}} \,\mathrm{d}C_s = 0, \quad t \in [0,T].$$

Hint: Consider the Lebesgue decomposition of $\mathbb{P} \otimes B$ with respect to $\mathbb{P} \otimes C$.

(c) Now let $S = (S_t)_{t \in [0,T]}$ be a continuous semimartingale with canonical decomposition $S = S_0 + M + A$, where $M \in \mathcal{M}_{0,\mathrm{loc}}^c$ and A is adapted, continuous, of finite variation and null at 0. Show that if S satisfies NA for 0-admissible strategies, then there exists a predictable process $H \in L(\langle M \rangle)$ such that

$$A_t = \int_0^t H_s \,\mathrm{d}\langle M \rangle_s, \quad t \in [0,T].$$

In other words, this means that S satisfies the structure condition (SC').

Hint: Write $A = A^+ - A^-$, where A^+ and A^- are the positive and the negative variation of A, respectively, use part (b) and argue by contradiction.

Exercise 2-5

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,1]}$. Denote by $(\mathscr{F}_t^W)_{t\in[0,1]}$ the natural (completed) filtration of W. Let $\mu = (\mu_t)_{t\in[0,1]}$ be a predictable process with $\int_0^1 |\mu_s| \, \mathrm{d}s < \infty$ $\mathbb{P}$-a.s. and $\int_0^t \mu_s^2 \, \mathrm{d}s < \infty$ $\mathbb{P}$-a.s. for all $t \in (0,1)$. Define the process $S = (S_t)_{t\in[0,1]}$ by the SDE

$$\mathrm{d}S_t = S_t(\mu_t \, \mathrm{d}t + \mathrm{d}W_t), \quad S_0 = s_0 > 0.$$

Moreover, define the process $Z = (Z_t)_{t\in[0,1]}$ by the SDE

$$\mathrm{d}Z_t = -Z_t \mu_t \, \mathrm{d}W_t, \quad Z_0 = 1. \tag{$*$}$$

(a) Show that $(*)$ has a unique strong solution, which is nonnegative. (Note that Z_1 can become 0.)

Hint: Use that $(*)$ has a unique strong solution on $[0,t]$ for fixed $t \in (0,1)$ and that this solution has an explicit formula. Paste these solutions together and use the supermartingale convergence theorem.

(b) Show that Z is the unique nonnegative local $\mathbb{P}$-martingale for the filtration $(\mathscr{F}_t^W)_{t\in[0,1]}$ with $Z_0 = 1$ such that ZS is also a local $\mathbb{P}$-martingale for the filtration $(\mathscr{F}_t^W)_{t\in[0,1]}$.

Hint: Use the product formula and Itô's representation theorem.

(c) Suppose that Z is a true martingale. Show that the process S satisfies NA.

Hint: Define $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_1$ by $\mathrm{d}\mathbb{Q} := Z_1 \, \mathrm{d}P$. Then show that if $\vartheta \bullet S_1 \geq 0$ $\mathbb{P}$-a.s. for $\vartheta \in \Theta_{\mathrm{adm}}$ then $\vartheta \bullet S \equiv 0$ under $\mathbb{Q}$ and use that $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_t$ for all $t \in [0,1)$.

(d) Show that S satisfies NFLVR if and only if Z is a true martingale with the additional property that $Z_1 > 0$ $\mathbb{P}$-a.s.

Hint: Use the fundamental theorem of asset pricing.

(e) Assume that $(\mu_t)_{t\in[0,1]}$ is given by $\mu_t := \frac{1}{\sqrt{1-t}} \mathbb{1}_{]\!]0,\tau]\!]}$, where $\tau = \inf\{t \in (0,1) : \widetilde{Z}_t = 2\} \wedge 1$ and where $\widetilde{Z} = (\widetilde{Z}_t)_{t\in[0,1]}$ is the unique strong solution of the SDE

$$\mathrm{d}\widetilde{Z}_t = \frac{-\widetilde{Z}_t}{\sqrt{1-t}} \, \mathrm{d}W_t, \quad \widetilde{Z}_0 = 1.$$

Using the above results, show that S satisfies NA but fails NFLVR.