ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 3

Exercise 3-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_k)_{k=0,\dots,T}, \mathbb{P})$ be a filtered probability space and $X = (X_k)_{k=0,\dots,T}$ an adapted process, which is null at 0.

(a) Suppose that X is a local martingale. Show that X is a true martingale if and only if $X_T^- \in L^1(\mathbb{P})$. *Hint:* The "$\Leftarrow$"-part is the difficult one. Divide the proof into two steps:

  (i) Show by backward induction first that $X_k^- \in L^1(\mathbb{P})$ and then that $X_k \in L^1(\mathbb{P})$ for $k = T-1, \dots, 0$. To this end, let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence and use that $(X^{\tau_n})^-$ is a submartingale for all $n \in \mathbb{N}$.

  (ii) Show that an integrable local martingale in discrete time is a martingale.

(b) Suppose that X is locally integrable. Show that X is a local martingale if and only if

$$\mathbb{E}\left[(\vartheta \bullet X)_T\right] \leq 0$$

for all predictable processes ϑ with $(\vartheta \bullet X)_T^- \in L^1(\mathbb{P})$.

Remark: Only the trivial "$\Rightarrow$"-direction of part (a) extends to continuous time. All other directions are *wrong* in continuous time.

Exercise 3-2

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

(a) Let $X = (X_t)_{t\geq 0}$ be a nonnegative supermartingale, and define the stopping time τ_0 by

$$\tau_0 := \inf\{t > 0 : X_t \wedge X_{t-} = 0\}.$$

Show that $X = 0$ on $[\![\tau_0, \infty]\!]$ $\mathbb{P}$-a.s.

Hint: For $n \in \mathbb{N}$, consider the stopping time $\tau_n := \inf\{t > 0 : X_t < 1/n\}$.

Remark: This result is known as the *minimum principle* for nonnegative supermartingales.

(b) Let $X = (X_t)_{t\geq 0}$ be a strictly positive local martingale with $X_0 = 1$. Show that there exists a unique local martingale $M = (M_t)_{t\geq 0}$ null at 0 such that $X = \mathcal{E}(M)$. M is called the *stochastic logarithm* of X and denoted by $\mathcal{L}(X)$.

Hint: Use part (a) and use without proof the fact that if Y is a local martingale and H is predictable and locally bounded, then $H \bullet Y$ is again a local martingale.

Exercise 3-3

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions.

(a) Let $N = (N_t)_{t\geq 0}$ be a local martingale null at 0 and $M = (M_t)_{t\geq 0}$ a *continuous* local martingale null at 0. Show that there exist $H \in L^2_{\mathrm{loc}}(M)$ and a local martingale $L = (L_t)_{t\geq 0}$ null at 0 such that

$$N = H \bullet M + L \quad \text{and} \quad [M, L] \equiv 0.$$

This is a generalised version of the *Kunita-Watanabe decomposition.*

Hint: Apply the usual Kunita-Watanabe decomposition to N^c.

(b) Let $S = (S_t)_{t\geq 0}$ be a continuous semimartingale and $S = S_0 + M + A$ its canonical decomposition. Prove that S satisfies the structure condition (SC), i.e., there exists $H \in L^2_{\mathrm{loc}}(M)$ such that

$$A_t = \int_0^t H_s \,\mathrm{d}\langle M\rangle_s, \quad t \geq 0,$$

if and only if there exists an *equivalent local martingale deflator* $Z = (Z_t)_{t\geq 0}$ for S, i.e., a strictly positive local $\mathbb{P}$-martingale Z with $Z_0 = 1$ such that ZS is a local $\mathbb{P}$-martingale.

Hint: For "$\Leftarrow$" use Exercise 3-2 (b) and part (a).

Exercise 3-4

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a Poisson process $N = (N_t)_{t\in[0,T]}$ with rate $\lambda > 0$. Denote by $(\mathscr{F}_t^N)_{t\in[0,T]}$ the natural (completed) filtration of N. Define the process $S = (S_t)_{t\in[0,T]}$ by $\mathrm{d}S_t = S_{t-}(\mu \,\mathrm{d}t + \frac{\sigma}{\sqrt{\lambda}}\,\mathrm{d}\widetilde{N}_t)$, $S_0 = s_0 > 0$, where $\mu \in \mathbb{R}$, $\sigma > 0$ and $\widetilde{N}$ denotes the compensated Poisson process.

(a) Show that S satisfies NFLVR if and only if $\mu < \sigma\sqrt{\lambda}$, and find an equivalent martingale measure $\mathbb{Q}^\lambda$ for S in that case.

Hint: Use Exercises 1-4 and 1-5.

Assume for the rest of the question that S satisfies NFLVR. Moreover, use without proof that the equivalent martingale measure $\mathbb{Q}^\lambda$ from part (a) is unique. For $\rho > 0$ denote by $\overline{\Psi}_\rho$ the tail distribution function of a Poisson random variable with parameter ρ, i.e., $\overline{\Psi}_\rho(x) := \mathbb{P}[X_\rho > x]$, where X_ρ has a Poisson distribution with parameter ρ.

(b) Show that the arbitrage-free price of a *cash-or-nothing call option* $\mathbb{1}_{\{S_T > K\}}$ with maturity T and strike $K > 0$ is given by

$$\overline{\Psi}_{(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

(c) Show that the arbitrage-free price of a *stock-or-nothing call option* $S_T\mathbb{1}_{\{S_T > K\}}$ with maturity T and strike $K > 0$ is given by

$$S_0\overline{\Psi}_{\left(1+\frac{\sigma}{\sqrt{\lambda}}\right)(\lambda - \frac{\mu}{\sigma}\sqrt{\lambda})T}\left(\frac{\log\frac{K}{S_0} + \left(\sigma\sqrt{\lambda} - \mu\right)T}{\log\left(1 + \frac{\sigma}{\sqrt{\lambda}}\right)}\right).$$

Hint: Define the measure $\widetilde{\mathbb{Q}}^\lambda \approx \mathbb{Q}^\lambda$ on $\mathscr{F}_T$ by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}^\lambda}{\mathrm{d}\mathbb{Q}^\lambda} := S_T/S_0$, and work under this measure.

(d) Derive the arbitrage-free price C_0^λ of a call option $(S_T - K)^+$ with maturity T and strike K. Moreover, show that

$$\lim_{\lambda\to\infty} C_0^\lambda = S_0\Phi\left(\frac{\log\frac{S_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{S_0}{K} - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right), \tag{1}$$

where Φ denotes the distribution function of a standard normal random variable. This means that for large λ, the arbitrage-free price in the Poisson model is very close to the Black-Scholes price with the same parameter σ.

Hint: Use that if X_ρ has a Poisson distribution with parameter ρ, then $\frac{X_\rho-\rho}{\sqrt{\rho}}$ converges weakly to a standard normal random variable for $\rho \to \infty$. Moreover, use the fact that if $(F_n)_{n\in\mathbb{N}}$ is a sequence of distribution functions converging *pointwise* to a *continuous* distribution function F, then the convergence is also *uniform*.

Exercise 3-5

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $R = (R_t)_{t\in[0,T]}$ be a jump diffusion of the form $R_t = \mu t + \sigma W_t + aN_t$, $t \in [0, T]$, where $\mu \in \mathbb{R}$ and $\sigma, a > 0$, $W = (W_t)_{t\in[0,T]}$ is a Brownian motion and $N = (N_t)_{t\in[0,T]}$ is an independent Poisson process with rate $\lambda > 0$. Suppose about the filtration that R is a Lévy process with respect to $(\mathscr{F}_t)_{t\in[0,T]}$. Define the process $S = (S_t)_{t\in[0,T]}$ by $\mathrm{d}S_t = S_{t-}\,\mathrm{d}R_t$, $S_0 = s_0 > 0$, i.e., $S = s_0\mathcal{E}(R)$.

(a) Show that for each $\ell > 0$ there exists $\mathbb{Q}^\ell \approx \mathbb{P}$ on $\mathscr{F}_T$ such that $R = \sigma W^{\mathbb{Q}^\ell} + a\widetilde{N}^{\mathbb{Q}^\ell}$, where $W^{\mathbb{Q}^\ell}$ is a $\mathbb{Q}^\ell$-Brownian motion and $\widetilde{N}^{\mathbb{Q}^\ell}$ is an independent compensated $\mathbb{Q}^\ell$-Poisson process with rate ℓ.

Hint: Use Girsanov's theorem and Exercise 1-4.

(b) Show that

$$\lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[\mathbb{1}_{\{S_T>K\}}] = 0.$$

Hint: Condition on $W^{\mathbb{Q}^\ell}$, use Chebyshev's inequality and dominated convergence.

(c) Show that

$$\lim_{\ell\to\infty} \mathbb{E}_{\mathbb{Q}^\ell}[S_T\mathbb{1}_{\{S_T\le K\}}] = 0.$$

Hint: For $\ell > 0$, define $\widetilde{\mathbb{Q}}^\ell \approx \mathbb{Q}^\ell$ on $\mathscr{F}_T$ by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}^\ell}{\mathrm{d}\mathbb{Q}^\ell} := S_T/S_0$, and work under this measure.

(d) Compute the superreplication prices of a call and a put option with strike $K > 0$ and maturity T, i.e., calculate $\Pi_s((S_T - K)^+)$ and $\Pi_s((K - S_T)^+)$.

Hint: Use parts (b) and (c).