ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 5

Exercise 5-1

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space with $\mathscr{F}_0$ $\mathbb{P}$-trivial and $(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, and let $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued semimartingale satisfying NFLVR. For $x > 0$, define the set $\mathcal{C}(x)$ as in the lecture. Moreover, let $U : (0,\infty) \to \mathbb{R}$ be an increasing and concave (utility) function, and suppose that $u(x) := \sup_{f\in\mathcal{C}(x)} \mathbb{E}[U(f)] < \infty$ for some (and hence all) $x > 0$. Set $U^+(\mathcal{C}(x)) := \{U^+(f) : f \in \mathcal{C}(x)\}$, and denote by $L^0_+(\mathscr{F}_T; [0,\infty])$ the space of all random variables taking values in $[0,\infty]$, endowed with the topology of convergence in probability.

(a) Fix $x > 0$. Show that $\mathcal{C}(x)$ is convex and closed in $L^0_+(\mathscr{F}_T; [0,\infty])$.

Hint: If $\mathcal{F}_0$ is trivial and $x > 0$, then for any $f \in L^0_+(\mathscr{F}_T) : f \in \mathcal{C}(x)$ if and only if $\mathbb{E}[fh] \le x$ for any $h \in \mathcal{D}(1)$.

(b) Fix $x > 0$. Suppose that that $U^+(\mathcal{C}(x))$ is uniformly integrable. Using only Lemma 6.2 in the lecture notes and part (a), show directly that there exists $f^* \in \mathcal{C}(x)$ such that $\mathbb{E}[U(f^*)] = u(x)$.

(c) Suppose now that there exist $a > 0$ and $b \in (0,1)$ such that $U^+(x) \le a(1 + x^b)$ for all $x > 0$ and an equivalent σ-martingale measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$ for S such that $\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^{-1}$ has moments of all orders. Fix $x > 0$, and show that $U^+(\mathcal{C}(x))$ is uniformly integrable.

Hint: By using the growth assumption on U^+, reduce the problem to showing that $\mathcal{C}(x)$ is bounded in L^p with $p > 1$ small enough. Then switch from $\mathbb{P}$- to $\mathbb{Q}$-expectations.

Exercise 5-2

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space with $\mathscr{F}_0$ $\mathbb{P}$-trivial and $(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and let $S = (S_t^1, \dots, S_t^d)_{t\in[0,T]}$ be an $\mathbb{R}^d$-valued semimartingale satisfying NFLVR. Assume that there exists a unique equivalent σ-martingale measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$, i.e., the market $(1, S)$ is complete. Let $U : (0,\infty) \to \mathbb{R}$ be a utility function as in the lecture. We assume that $u(x) < \infty$ for some (and hence all) $x \in (0,\infty)$. We do **not** assume, however, that $AE_{+\infty}(U) < 1$. Define the functions $J, I, u, j : (0,\infty) \to \mathbb{R} \cup \{\infty\}$ and the sets $\mathcal{C}(x)$ and $\mathcal{D}(z)$, $x, z > 0$, as in the lecture.

(a) Fix $z > 0$. Show that

$$h \leq z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \quad \mathbb{P}\text{-a.s. for all } h \in \mathcal{D}(z),$$

where $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ denotes the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathscr{F}_T$. Deduce that

$$j(z) = \inf_{h\in\mathcal{D}(z)} \mathbb{E}[J(h)] = \mathbb{E}\left[J\left(z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right],$$

where $\mathbb{E}[J(h)] := +\infty$ if $J^+(h) \notin L^1(\mathbb{P})$.

Hint: Suppose that there is $h \in \mathcal{D}(z)$ such that $A := \{h > z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\}$ has $\mathbb{P}[A] > 0$. Set $a := \mathbb{Q}[A]$, and use that completeness of S is equivalent to the *predictable representation property* of S under $\mathbb{Q}$ to deduce that there exists $H \in L(S)$ such that $\mathbb{1}_A = a + H \bullet S_T$.

(b) Let $z_0 := \inf\{z > 0 : j(z) < \infty\}$. Show that the function j is in $C^1(z_0, \infty)$ and satisfies

$$j'(z) = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} J'\left(z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right], \quad z \in (z_0, \infty).$$

Hint: Apply the fundamental theorem of calculus to the function $z \mapsto J\left(z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ and take expectations.

(c) Set $x_0 := \lim_{z\downarrow\downarrow z_0} -j'(z)$. Fix $x \in (0, x_0)$. Let $z_x \in (z_0, \infty)$ be the unique number such that $-j'(z_x) = x$. Show that $f^* := I\left(z_x\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)$ is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} \mathbb{E}[U(f)].$$

Hint: Show that $f^* \in \mathcal{C}(x)$ using the hint in Ex 5-1 a) and part (a). Then use a Taylor expansion and the strict concavity of U in $(0,\infty)$ to argue that we have $\mathbb{E}[U(f) - U(f^*)] \leq 0$ for all $f \in \mathcal{C}(x)$ and that the inequality is an equality if and only if $f = f^*$ $\mathbb{P}$-a.s.

Exercise 5-3

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,T]}$. Denote by $(\mathscr{F}_t^W)_{t\in[0,T]}$ the natural (completed) filtration of W. Let $\sigma > 0$ and $\mu, r \in \mathbb{R}$. Consider the *undiscounted* Black-Scholes market $(\widetilde{S}^0, \widetilde{S}^1) = (\widetilde{S}_t^0, \widetilde{S}_t^1)_{t\in[0,T]}$ given by the SDEs

$$\mathrm{d}\widetilde{S}_t^0 = r\widetilde{S}_t^0\,\mathrm{d}t, \quad \widetilde{S}_0^0 = 1, \quad \text{and} \quad \mathrm{d}\widetilde{S}_t^1 = \widetilde{S}_t^1(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \quad \widetilde{S}_0^1 = s > 0.$$

Denote by $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$ the discounted stock price. Let $U : (0,\infty) \to \mathbb{R}$ be defined by $U(x) = \frac{1}{\gamma}x^\gamma$, where $\gamma \in (-\infty, 1) \setminus \{0\}$. We consider the *Merton problem* of maximising expected utility from final wealth (in units of $\widetilde{S}^0$), where we use the notation from the lecture.

(a) Using Exercise 5-2 (a) show that

$$j(z) = \frac{1-\gamma}{\gamma} z^{-\frac{\gamma}{1-\gamma}} \exp\left(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad z \in (0,\infty).$$

(b) Using Exercise 5-2 (c) show that $f_x^* := x\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_T$, where $R = (R_t)_{t\in[0,T]}$ is given by $R_t = W_t + \frac{\mu-r}{\sigma}t$, is the unique solution to the primal problem

$$u(x) = \sup_{f\in\mathcal{C}(x)} \mathbb{E}[U(f)], \quad x \in (0,\infty).$$

(c) Deduce that $f_x^* = V_T(x,\vartheta^x)$, where $\vartheta^x = (\vartheta_t^x)_{t\in[0,T]}$ is given by

$$\vartheta_t^x = \frac{x}{S_t}\frac{1}{1-\gamma}\frac{\mu-r}{\sigma^2}\mathcal{E}\left(\frac{1}{1-\gamma}\frac{\mu-r}{\sigma}R\right)_t, \quad x \in (0,\infty),$$

and show that

$$u(x) = \frac{x^\gamma}{\gamma}\exp\left(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2}T\right), \quad x \in (0,\infty).$$

Exercise 5-4

Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ be a filtered probability space satisfying the usual conditions and $S = (S_t)_{t\in[0,T]}$ a real-valued *continuous* semimartingale. The set of all (tradeable) *numéraires* is defined by

$$\mathcal{N} := \{1 + \vartheta \bullet S : \vartheta \in L(S) \text{ and } 1 + \vartheta \bullet S > 0 \ \ \mathbb{P}\text{-a.s.}\}.$$

A numéraire $N^* \in \mathcal{N}$ is called a *numéraire portfolio* if $\frac{N}{N^*}$ is a $\mathbb{P}$-supermartingale for all $N \in \mathcal{N}$.

(a) Show that if a numéraire portfolio exists, then it is unique.

(b) Show that if S satisfies the structure condition (SC), i.e., $S = S_0 + M + \lambda \bullet \langle M\rangle$, where $M \in \mathcal{H}^{2,c}_{0,\text{loc}}$ and $\lambda \in L^2_{\text{loc}}(M)$, then the numéraire portfolio exists and is given by

$$N^* = \mathcal{E}(\lambda \bullet S).$$

Hint: Show that $Z := 1/N^*$ is an equivalent local martingale deflator for S, i.e., a strictly positive local $\mathbb{P}$-martingale with $Z_0 = 1$ such that ZS is a local $\mathbb{P}$-martingale. Deduce that $Z(\vartheta \bullet S)$ is a local $\mathbb{P}$-martingale for all $\vartheta \in L(S)$ by applying the product formula twice.

Suppose for the rest of the question that the numéraire portfolio N^* exists, and let $\vartheta^* \in L(S)$ be such that $N^* = 1 + \vartheta^* \bullet S$.

(c) Set $\widehat{S}^1 := \frac{S}{N^*}$ and $\widehat{S}^2 := \frac{1}{N^*}$. Show that for $i = 1,2$, $\mathbb{P}$ is a separating measure for $\widehat{S}^i$.

Hint: Fix $i \in \{1,2\}$. For $\widehat{\vartheta}^i \in \Theta_{\text{adm}}(\widehat{S}^i)$, find $a^i > 0$ and $\widetilde{N}^i \in \mathcal{N}$ such that $\frac{a^i + \widehat{\vartheta}^i \bullet \widehat{S}^i}{a^i} = \frac{\widetilde{N}^i}{N^*}$ and apply the definition of the numéraire portfolio. To this end, show in a first step by using the product formula twice that if $\widehat{\vartheta}^i \in L(\widehat{S}^i)$, then there is $\vartheta^i \in L(S)$ such that $\widehat{\vartheta}^i \bullet \widehat{S}^i = \frac{\vartheta^i \bullet S}{N^*}$.

Remark One can also show that $\mathbb{P}$ is a separating measure for $(\widehat{S}^1,\widehat{S}^2)$ but this requires vector stochastic integration.

(d) Using Exercises 2-2 (a) and 3-3 (b) deduce that S satisfies the structure condition (SC).

Exercise 5-5

Consider the setup and notation of Exercise 5-4. Set

$$\ell := \sup_{N\in\mathcal{N}} \mathbb{E}\left[\log N_T\right] \in [0,\infty], \tag{$*$}$$

where $\mathbb{E}\left[\log N_T\right] := -\infty$ if $\log^- N_T \notin L^1(\mathbb{P})$. If $\ell < \infty$, then the unique optimiser of $(*)$, if it exists, is denoted by $N^{\log}$ and called the *growth-optimal portfolio.*

(a) Show that if the numéraire portfolio N^* exists and $\ell < \infty$, then the growth optimal portfolio exists and $N^{\log} = N^*$.

Hint: Use that $\log x \geq \log y + 1 - y/x$ for all $x, y > 0$, because log is concave.

Suppose for the rest of the question that $\ell < \infty$ and that the growth optimal portfolio $N^{\log}$ exists.

(b) Fix $N \in \mathcal{N}$. Show that

$$\mathbb{E}\left[\frac{N_T}{N_T^{\log}}\right] \leq 1.$$

Hint: For $\epsilon \in (0,1)$, set $N^\epsilon := \epsilon N + (1-\epsilon)N^{\log} \in \mathcal{N}$ and show that $\mathbb{E}\left[\frac{N_T^{\log} - N_T^\epsilon}{N_T^\epsilon}\right] \geq 0$. Then let $\epsilon \downarrow 0$.

(c) Let $0 \leq s \leq T$, $A \in \mathscr{F}_s$ and $N^1, N^2 \in \mathcal{N}$ with associated ϑ^1, ϑ^2. Then the strategy of *switching from N^1 to N^2 at time s on A* is given by

$$\widetilde{\vartheta} := \mathbb{1}_{[\![0,s]\!]}\vartheta^1 + \mathbb{1}_{]\!]s,T]\!]}\left(\mathbb{1}_A \frac{N_s^1}{N_s^2}\vartheta^2 + \mathbb{1}_{A^c}\vartheta^1\right).$$

Show that $\widetilde{\vartheta} \in L(S)$ and $\widetilde{N} := 1 + \widetilde{\vartheta} \bullet S \in \mathcal{N}$ with

$$\widetilde{N} = \mathbb{1}_{[\![0,s]\!]}N^1 + \mathbb{1}_{]\!]s,T]\!]}\left(\mathbb{1}_A \frac{N_s^1}{N_s^2}N^2 + \mathbb{1}_{A^c}N^1\right).$$

(d) Deduce that the numéraire portfolio exists and that $N^* = N^{\log}$.

Hint: Fix $N \in \mathcal{N}$ and $0 \leq s < t \leq T$. Set $A := \{\mathbb{E}[N_t/N_t^{\log} \mid \mathscr{F}_s] > N_s/N_s^{\log}\}$ and consider the numéraire $\widehat{N}$ corresponding to the strategy of switching first from $N^{\log}$ to N at time s on A and then back to $N^{\log}$ at time t on A. Then apply part (b) to $\widehat{N}$.