ETH Zürich HS 2013
D-MATH
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Mathematical Finance

Exercise Sheet 6

Exercise 6-1

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $W = (W_t)_{t\in[0,T]}$. Denote by $(\mathscr{F}_t^W)_{t\in[0,T]}$ the natural (completed) filtration of W. Let $\sigma > 0$ and $\mu, r \in \mathbb{R}$. Consider the *undiscounted* Black-Scholes market $(\widetilde{S}^0, \widetilde{S}^1) = (\widetilde{S}^0_t, \widetilde{S}^1_t)_{t\in[0,T]}$ given by the SDEs

$$\mathrm{d}\widetilde{S}^0_t = r\widetilde{S}^0_t\,\mathrm{d}t, \quad \widetilde{S}^0_0 = 1, \quad \text{and} \quad \mathrm{d}\widetilde{S}^1_t = \widetilde{S}^1_t(\mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t), \quad \widetilde{S}^1_0 = s > 0.$$

Denote by $S^1 := \frac{\widetilde{S}^1}{\widetilde{S}^0}$ the discounted stock price and by $R = (R_t)_{t\in[0,T]}$ the *returns process of* S^1, i.e., $R_t = (\mu - r)t + \sigma W_t$, $t \in [0,T]$. Let $U : \mathbb{R} \to \mathbb{R}$ be given by $U(x) = -\exp(-\alpha x)$, where $\alpha > 0$. Set

$$\mathcal{A} := \{\vartheta \in L(S^1) : \vartheta S^1 \geq -a \text{ for some } a > 0\},$$

i.e., the set of all strategies whose *risky position* ϑS^1 is uniformly bounded from below. For $x \in \mathbb{R}$, we consider the problem of maximising expected utility from final wealth (in units of $\widetilde{S}^0$), i.e., we seek $\vartheta^*_x \in \mathcal{A}$ such that

$$\mathbb{E}[U(V_T(x, \vartheta^*_x))] = \sup_{\vartheta\in\mathcal{A}} \mathbb{E}[U(V_T(x, \vartheta))] =: u(x) < \infty.$$

It turns out to be convenient to reformulate this problem a bit. To this end, set

$$\mathcal{A}' := \{\varphi \in L(R) : \varphi \geq -a \text{ for some } a > 0\} \quad \text{and} \quad V'(x, \varphi) := x + \varphi \bullet R, \quad x \in \mathbb{R}, \varphi \in L(R).$$

Then $u(x) = \sup_{\varphi\in\mathcal{A}'} \mathbb{E}[U(V'_T(x, \varphi))]$ and φ^*_x is an optimiser of the reformulated problem if and only if $\vartheta^*_x := \frac{\varphi^*_x}{S^1}$ is an optimiser of the original problem. Finally, to use the tools from stochastic optimal control, set

$$\begin{aligned} \mathcal{A}'(t, \varphi) &:= \{\psi \in \mathcal{A}' : \psi = \varphi \text{ on } [\![0, t]\!]\}, && t \in [0,T], \varphi \in \mathcal{A}', \\ J_{t,x}(\varphi) &:= \operatorname*{ess\,sup}_{\psi\in\mathcal{A}'(t,\varphi)} \mathbb{E}[U(V'_T(x, \psi)) \,|\, \mathscr{F}_t], && t \in [0,T], x \in \mathbb{R}, \varphi \in \mathcal{A}'. \end{aligned}$$

(a) Make the *ansatz* that there exists a real-valued function v in $C^{1,2}([0,T)\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ such that $J_{t,x}(\varphi) = v(t, V'_t(x, \varphi))$. Argue that v should satisfy the HJB equation

$$\begin{aligned} v_t(t, x) + \sup_{\rho\in\mathbb{R}} \left(\rho(\mu - r)v_x(t, x) + \frac{1}{2}\rho^2\sigma^2 v_{xx}(t, x)\right) &= 0, \quad t \in [0,T), x \in \mathbb{R}, \\ v(T, x) &= U(x), \quad x \in \mathbb{R}. \end{aligned} \tag{$*$}$$

Hint: Apply Itô's formula and the martingale optimality principle.

(b) *Assume* that $v(t,\cdot)$ is strictly concave for all $t \in [0,T)$. Use this to find a real-valued function ρ^* in $C^0([0,T)\times\mathbb{R})$ such that for each $t \in [0,T)$, $x \in \mathbb{R}$,

$$\sup_{\rho\in\mathbb{R}}\left(\rho(\mu-r)v_x(t,x)+\frac{1}{2}\rho^2\sigma^2 v_{xx}(t,x)\right)=\rho^*(t,x)(\mu-r)v_x(t,x)+\frac{1}{2}(\rho^*(t,x))^2\sigma^2 v_{xx}(t,x)$$

and deduce that under the strict concavity assumption, $(*)$ is equivalent to

$$0 = v_t - \frac{(\mu-r)^2}{2\sigma^2}\frac{(v_x)^2}{v_{xx}} \quad \text{in } [0,T)\times\mathbb{R} \quad \text{and} \quad v(T,\cdot)=U(\cdot) \quad \text{on } \mathbb{R}. \qquad (**)$$

(c) Show that the ansatz of part (a) implies that $v(t,x)=\exp(-ax)w(t)$ for some real-valued function w in $C^1([0,T))\cap C^0([0,T])$. Use this to solve the PDE $(**)$ explicitly, and deduce that $\rho^*(t,x)=\frac{\mu-r}{\alpha\sigma^2}$, $t\in[0,T)$, $x\in\mathbb{R}$.

(d) For $x \in \mathbb{R}$, set $\varphi^*_x := \varphi^* := \frac{\mu-r}{\alpha\sigma^2}$. Using without proof that indeed $J_{t,x}(\varphi)=v(t,V'_t(x,\varphi))$ for all $t\in[0,T]$, $x\in\mathbb{R}$ and $\varphi\in\mathcal{A}'$, show that φ^* is an optimiser of the reformulated problem, and thus $\vartheta^*_x := \vartheta^* = \frac{\varphi^*}{S^1}$ is an optimiser of the original problem.

Hint: Apply the the martingale optimality principle.

Exercise 6-2

Let $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ be a filtered probability space and $W=(W_t)_{t\in[0,t]}$ a Brownian motion for $(\mathscr{F}_t)_{t\in[0,T]}$. Let $(1,S)=(1,S_t)_{t\in[0,T]}$ be a *Bachelier market*, i.e., $S_t=S_0+\mu t+\sigma W_t$, $t\in[0,T]$, where $\sigma>0$, $\mu\in\mathbb{R}$ and $S_0=s>0$. Then $S=S_0+M+\lambda\bullet\langle M\rangle$, where $M=\sigma W$ and $\lambda=\frac{\mu}{\sigma^2}$. Set $\Theta=L^2(M)$. We consider the particular problem of finding the optimal strategy $\vartheta^*\in\Theta$ such that

$$\mathbb{E}[(1-V_T(0,\vartheta^*))^2]=\inf_{\vartheta\in\Theta}\mathbb{E}[(1-V_T(0,\vartheta))^2].$$

For $t\in[0,T]$ and $y\in L^2(\mathscr{F}_t,\mathbb{P})$, define $J^1_t(y)$ as in the lecture and recall that

$$J^1_t(y)=a_t y^2-2b_t y+c_t, \quad t\in[0,T],$$

where $a=(a_t)_{t\in[0,T]}$, $b=(b_t)_{t\in[0,T]}$ and $c=(c_t)_{t\in[0,T]}$ are semimartingales and satisfy the BSDEs

$$\mathrm{d}a_t = a_{t-}\left(\lambda+\frac{\nu^a_t}{a_{t-}}\right)^2\mathrm{d}\langle M\rangle_t+\nu^a_t\,\mathrm{d}M_t+\mathrm{d}L^a_t, \qquad a_T=1; \qquad \text{(A)}$$

$$\mathrm{d}b_t = \left(\lambda+\frac{\nu^a_t}{a_{t-}}\right)(\lambda b_{t-}+\nu^b_t)\,\mathrm{d}\langle M\rangle_t+\nu^b_t\,\mathrm{d}M_t+\mathrm{d}L^b_t, \qquad b_T=1; \qquad \text{(B)}$$

$$\mathrm{d}c_t = \frac{(\lambda b_{t-}+\nu^b_t)^2}{a_{t-}}\,\mathrm{d}\langle M\rangle_t+\mathrm{d}N^c_t, \qquad c_T=1, \qquad \text{(C)}$$

where $\nu^a,\nu^b\in L^2_{\mathrm{loc}}(M)$, L^a,L^b are local $\mathbb{P}$-martingales strongly orthogonal to M, and N^c is a local $\mathbb{P}$-martingale.

(a) Find a strong solution $(a,\nu^a,L^a,b,\nu^b,L^b,c,N^c)$ for the BSDEs (A) – (C).

Hint: Using that $\langle M\rangle$ is *deterministic*, try to find a solution such that a, b and c are deterministic. Moreover, first solve (A) and (B) and then (C).

(b) Find a strong solution X^* to the SDE

$$\mathrm{d}X^*_t=\left(\frac{\lambda b_{t-}+\nu^b_t}{a_{t-}}-X^*_t\left(\lambda+\frac{\nu^a_t}{a_{t-}}\right)\right)\mathrm{d}S_t, \quad X^*_0=0,$$

and deduce that $X^*=\vartheta^*\bullet S$, where $\vartheta^*_t=\lambda\mathcal{E}(-\lambda\bullet S)$, $t\in[0,T]$.

(c) Show that $\vartheta^* \in \Theta = L^2(M)$ and that it is indeed optimal.

Hint: For the first claim, argue that it suffices to show that $\lambda\mathcal{E}(-\lambda M) \in L^2(M)$. For the second claim, use the martingale optimality principle and calculate $J_t^1(V_t(0, \vartheta^*))$, $t \in [0, T]$, explicitly.

Exercise 6-3

Consider the same setup as in Exercise 6-2. We now consider an investor who wants to invest into the market $(1, S)$ by choosing a strategy $\vartheta \in \Theta = L^2(M)$. For $\vartheta \in \Theta$, denote by $\alpha_\vartheta := \mathbb{E}[\vartheta \bullet S_T]$ the *return* and by $\beta_\vartheta := \sqrt{\mathrm{Var}[\vartheta \bullet S_T]}$ the *volatility* of ϑ.

(a) Let $\vartheta^* = \lambda\mathcal{E}(-\lambda \bullet S)$ be as in Exercise 6-2. Set $\alpha^* := \alpha_{\vartheta^*}$ and $\beta^* := \beta_{\vartheta^*}$. Show that

$$\beta_\vartheta \geq \beta^* \quad \text{for all } \vartheta \in \Theta \text{ satisfying } \alpha_\vartheta = \alpha^*.$$

(b) Show that

$$\alpha_\vartheta \leq \beta_\vartheta \sqrt{\exp\left(\frac{\mu^2}{\sigma^2}T\right) - 1} \quad \text{for all } \vartheta \in \Theta.$$

Moreover, show that if $\mu \neq 0$, then for each $\alpha > 0$, there exists ϑ_α such that

$$\alpha_{\vartheta_\alpha} = \alpha \quad \text{and} \quad \frac{\alpha_{\vartheta_\alpha}}{\beta_{\vartheta_\alpha}} = \sqrt{\exp\left(\frac{\mu^2}{\sigma^2}T\right) - 1}.$$

Hint: Show that $\frac{\alpha^*}{\beta^*} := \sqrt{\exp\left(\frac{\mu^2}{\sigma^2}T\right) - 1}$.

Exercise 6-4

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathcal{Y} := L^\infty(\mathscr{F}, \mathbb{P})$ and $\rho : \mathcal{Y} \to \mathbb{R}$ a map. We consider different properties/axioms for ρ:

- **Monotonicity (M):** $\rho(Y^1) \leq \rho(Y^2)$ for all $Y^1, Y^2 \in \mathcal{Y}$ with $Y^1 \geq Y^2$ $\mathbb{P}$-a.s.
- **Translation invariance (T):** $\rho(Y + c) = \rho(Y) - c$ for all $Y \in \mathcal{Y}$ and $c \in \mathbb{R}$.
- **Subadditivity (S):** $\rho(Y^1 + Y^2) \leq \rho(Y^1) + \rho(Y^2)$ for all $Y^1, Y^2 \in \mathcal{Y}$.
- **Positive homogeneity (PH)** $\rho(\lambda Y) = \lambda\rho(Y)$ for all $Y \in \mathcal{Y}$ and $\lambda \geq 0$.
- **Convexity (C):** $\rho(\lambda Y^1 + (1-\lambda)Y^2) \leq \lambda\rho(Y^1) + (1-\lambda)\rho(Y^2)$ for all $Y^1, Y^2 \in \mathcal{Y}$ and $\lambda \in [0, 1]$.
- **Quasi-convexity (QC):** $\rho(\lambda Y^1 + (1-\lambda)Y^2) \leq \max(\rho(Y^1), \rho(Y^2))$ for all $Y^1, Y^2 \in \mathcal{Y}$ and $\lambda \in [0, 1]$.

(a) Show that if ρ satisfies (M) and (T), then ρ is Lipschitz continuous with respect to $\|\cdot\|_{L^\infty}$.

(b) Show that if ρ satisfies (T), then it satisfies (C) if and only if it satisfies (QC).

Hint: For "$\Leftarrow$", show that $\mathcal{A}_\rho := \{Y \in \mathcal{Y} : \rho(Y) \leq 0\}$ is a convex set.

(c) Show that if ρ satisfies (M), (T), (S) and (C) and $\rho(0) = 0$, then it also satisfies (PH).

Hint: First, using (S) and (C) show by induction that $\rho(\lambda Y) = \lambda\rho(Y)$ for all $Y \in \mathcal{Y}$ and all *rational* $\lambda > 0$. Then, use part (a).

Exercise 6-5

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathcal{Y} := L^1(\mathscr{F}, \mathbb{P})$ and $\alpha \in (0,1)$ a fixed sensitivity parameter. We consider the following maps $\mathcal{Y} \to \mathbb{R}$:

- **Value at Risk (**VaR_α**):** $\mathrm{VaR}_\alpha(Y) := \sup\{y : \mathbb{P}[Y \leq -y] > \alpha\}$.
- **Average Value at Risk (**AVaR_α**):** $\mathrm{AVaR}_\alpha(Y) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_u(Y)\,\mathrm{d}u$.
- **Tail Conditional Expectation (**TCE_α**):** $\mathrm{TCE}_\alpha(Y) := \mathbb{E}_{\mathbb{P}}[-Y \mid -Y \geq \mathrm{VaR}_\alpha(Y)]$.
- **Worst Conditional Expectation (**WCE_α**):**
  $\mathrm{WCE}_\alpha(Y) := \sup\{\mathbb{E}_{\mathbb{P}}[-Y \mid A] : A \in \mathscr{F} \text{ with } P[A] > \alpha\}$.

One can show that AVaR_α admits a dual characterisation

$$\mathrm{AVaR}_\alpha(Y) = \sup_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}_{\mathbb{Q}}[-Y], \quad Y \in \mathcal{Y},$$

where

$$\mathcal{Q}_\alpha := \left\{\mathbb{Q} \in \mathcal{M}_1^a : \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \leq \frac{1}{\alpha} \ \mathbb{P}\text{-a.s.}\right\}.$$

Moreover, by arguing as in the lecture, one can show that both AVaR_α and WCE_α are coherent risk measures on $\mathcal{Y}$.

(a) Show that for all $Y \in \mathcal{Y}$,

$$\mathrm{AVaR}_\alpha(Y) \geq \mathrm{WCE}_\alpha(Y) \geq \mathrm{TCE}_\alpha(Y) \geq \mathrm{VaR}_\alpha(Y),$$

and that the first two inequalities are equalities in case that Y has a continuous distribution.

Hint: For the first inequality, use the dual characterisation of AVaR_α. For the second inequality, argue that $\mathbb{P}[-Y \geq \mathrm{VaR}_\alpha(Y) - \epsilon] > \alpha$ for all $\epsilon > 0$. For the last statement, use without proof that if Y has a continuous distribution F_Y, then there exists a random variable U which is uniformly distributed on $(0,1)$ such that $Y = q_Y^+(U)$ $\mathbb{P}$-a.s., where $q_Y^+(u) := \inf\{y \in \mathbb{R} : F_Y(y) > u\}$, $u \in (0,1)$, is the *upper quantile function* of Y.

(b) Calculate $\mathrm{VaR}_\alpha(-Y)$ and $\mathrm{AVaR}_\alpha(-Y)$, where Y is

(i) exponentially distributed with rate parameter $\lambda > 0$,

(ii) Pareto distributed with parameter $\beta > 1$, i.e.,

$$\mathbb{P}[Y \geq y] = \begin{cases} y^{-\beta}, & y \geq 1, \\ 1, & y < 1, \end{cases}$$

(iii) lognormally distributed with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$.