ETH Zürich HS 2013
D-MATH
Prof. J. Teichmann

Mathematical Finance

Solution Sheet 2

Solution 2-1

(a) Define

$$k^* := \min\Big\{k \in \{1, \dots, N\} : G_{\tau_k}(\vartheta) \in L^0_+ \setminus \{0\}\Big\}, \tag{1}$$

and set $\sigma_0 := \tau_{k^*-1}$ and $\sigma_1 := \tau_{k^*}$. Moreover, set

$$h := \begin{cases} h^{k^*} & \text{if } \mathbb{P}[G_{\tau_{k^*-1}}(\vartheta) = 0] = 1, \\ h^{k^*} \mathbb{1}_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} & \text{if } \mathbb{P}[G_{\tau_{k^*-1}}(\vartheta) = 0] < 1. \end{cases} \tag{2}$$

Note that $\mathbb{P}[G_{\tau_{k^*-1}}(\vartheta) < 0] > 0$ in the second case by the definition of k^*. We claim that $\vartheta^* := h\mathbb{1}_{]\!]\sigma_0,\sigma_1]\!]} \in \mathbf{b}\mathcal{E}$ is an arbitrage opportunity. Indeed, in the first case,

$$G_T(\vartheta^*) = G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta) = G_{\tau_{k^*}}(\vartheta) \in L^0_+ \setminus \{0\}, \tag{3}$$

and in the second case,

$$\begin{aligned} G_T(\vartheta^*) &= \big(G_{\tau_{k^*}}(\vartheta) - G_{\tau_{k^*-1}}(\vartheta)\big)\mathbb{1}_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} \\ &\geq -G_{\tau_{k^*-1}}(\vartheta)\mathbb{1}_{\{G_{\tau_{k^*-1}}(\vartheta)<0\}} \in L^0_+ \setminus \{0\}. \end{aligned} \tag{4}$$

(b) Let $a > 0$ be such that $G(\vartheta) \geq -a$ $\mathbb{P}$-a.s. By right-continuity of the paths of $G(\vartheta)$, it suffices to show $G_t(\vartheta) \geq -c$ $\mathbb{P}$-a.s. for all $t \in [0, T)$. Seeking a contradiction, assume there is $t \in [0, T)$ such that $\mathbb{P}[G_t(\vartheta) < -c] > 0$. But then $\vartheta^* := \vartheta\mathbb{1}_{\{G_t(\vartheta)<-c\}\times(t,T]}$ is predictable, in $L(S)$ and satisfies

$$\begin{aligned} G(\vartheta^*) &= (G(\vartheta) - G_t(\vartheta))\mathbb{1}_{\{G_t(\vartheta)<-c\}\times(t,T]} \geq -a + c, \\ G_T(\vartheta^*)_T &= (G_T(\vartheta) - G_t(\vartheta))\mathbb{1}_{\{G_t(\vartheta)<-c\}} \geq (-c - G_t(\vartheta))\mathbb{1}_{\{G_t(\vartheta)<-c\}} \end{aligned} \tag{5}$$

But this shows both that ϑ^* is admissible and that S fails NA, in contradiction to the hypothesis.

Solution 2-2

(a) First, assume that S is bounded. Note that then every simple strategy is admissible. Moreover, S is a uniformly integrable $\mathbb{Q}$-martingale if and only if $\mathbb{E}_{\mathbb{Q}}[S_\tau - S_0] = 0$ for all stopping times (taking values in $[0,T]$). So let τ be an arbitrary stopping time, and consider the simple strategies $\vartheta := \mathbb{1}_{]\!]0,\tau]\!]}$ and $-\vartheta$. Using that $\mathbb{Q}$ is an equivalent separating measure for S then gives

$$0 \geq \mathbb{E}_{\mathbb{Q}}[\vartheta \bullet S_T] = \mathbb{E}_{\mathbb{Q}}[S_\tau - S_0] = -\mathbb{E}_{\mathbb{Q}}[-(S_\tau - S_0)] = -\mathbb{E}_{\mathbb{Q}}[(-\vartheta) \bullet S_T] \geq 0. \quad (6)$$

Next, consider the case that S is locally bounded. Then there exists an increasing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ taking values in $[0,T]$ with $\lim_{n\to\infty} \mathbb{P}[\sigma_n = T] = 1$ such that S^{σ_n} is bounded for all $n \in \mathbb{N}$. It suffices to show that for each $n \in \mathbb{N}$, S^{σ_n} is a uniformly integrable $\mathbb{Q}$-martingale. To this end, fix $n \in \mathbb{N}$. It suffices to show that for each stopping time τ with $\tau \leq \sigma_n$ $\mathbb{P}$-a.s., $\mathbb{E}_{\mathbb{Q}}[S_\tau - S_0] = 0$. So let τ be such a stopping time, and consider as above the simple strategies $\vartheta := \mathbb{1}_{]\!]0,\tau]\!]}$ and $-\vartheta$. Then both strategies are admissible since S is bounded on $[\![0,\sigma_n]\!]$ and $\tau \leq \sigma_n$ $\mathbb{P}$-a.s., and the same argument as in the first step gives $\mathbb{E}_{\mathbb{Q}}[S_\tau - S_0] = 0$.

(b) By assumption, there exist a strictly positive predictable process $\psi = (\psi_t)_{t\in[0,T]}$, an $\mathbb{R}^d$-valued (local) $\mathbb{Q}$-martingale M, and an $\mathbb{R}^d$-valued $\mathscr{F}_0$-measurable random vector S_0 such that $S = S_0 + \psi \bullet M$. Let $\vartheta \in \Theta_{\text{adm}}$. Then by the associativity of the stochastic integral, $G(\vartheta) = \vartheta \bullet S = (\vartheta\psi) \bullet M$. Moreover, as $(\vartheta\psi) \bullet M$ is uniformly bounded from below by admissibility, it is a local $\mathbb{Q}$-martingale by the Ansel-Stricker theorem. By Fatou's lemma, it is then also a $\mathbb{Q}$-supermartingale, and hence

$$\mathbb{E}_{\mathbb{Q}}[G_T(\vartheta)] \leq \mathbb{E}_{\mathbb{Q}}[G_0(\vartheta)] = 0. \quad (7)$$

(c) First, since $\mathscr{F}_t$ is $\mathbb{P}$-trivial for all $t \in [0,T)$, a process $\xi = (\xi_t)_{t\in[0,T]}$ is adapted if and only if it is deterministic on $[0,T)$ and ξ_T is $\sigma(X)$-measurable. In particular, all left-continuous and adapted processes for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ are deterministic, and by a monotone class argument, the same is true for all predictable processes.

Next, if $\vartheta \in L(S)$ is arbitrary, then

$$G_T(\vartheta) = \vartheta \bullet S_T = \vartheta \bullet S_{T-} + \vartheta_T \Delta S_T = \lim_{t\uparrow T} \vartheta \bullet S_t + \vartheta_T X = 0 + \vartheta_T X = \vartheta_T X. \quad (8)$$

Since ϑ_T is deterministic and X normally distributed, it follows that $\vartheta \in \Theta_{\text{adm}}$ if and only if $\vartheta_T = 0$. Thus, we may conclude that $G_T(\vartheta) = 0$ for all $\vartheta \in \Theta_{\text{adm}}$. Therefore the condition

$$\mathbb{E}_{\mathbb{Q}}[G_T(\vartheta)] \leq 0 \quad \text{for all } \vartheta \in \Theta_{\text{adm}}$$

is trivially satisfied for each probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$. In particular, $\mathbb{P}$ itself is a separating measure.

Finally if $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_T$ is an equivalent probability measure, by the first step (whose results remain unchanged by an equivalent change of measure), $M = (M_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale null at 0 for the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ if and only if M_T is $\sigma(X)$-measurable, $\mathbb{Q}$-integrable with mean 0 and $M_t = 0$ for all $t \in [0,T)$. Moreover, if $\psi \in L^{\mathbb{Q}}(M)$, then as M is constant and equal to 0 on $[0,T)$,

$$\psi \bullet M_t = \begin{cases} 0 & \text{for } t < T, \\ \psi_T M_T & \text{for } t = T. \end{cases} \quad (9)$$

Note that as ψ_T is constant, $\psi \bullet M$ is a true $\mathbb{Q}$-martingale, and therefore $\mathbb{Q}$ is a equivalent σ-martingale measure for S if and only if it is an equivalent martingale measure. Since $\mathbb{E}[S_T] = \mu \neq 0$, $\mathbb{P}$ is not a martingale measure and hence also not a σ-martingale measure.

Solution 2-3

(a) Define the measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \frac{1}{1+\|S_1\|} \Big/ \mathbb{E}\left[\frac{1}{1+\|S_1\|}\right], \tag{10}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Then $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is bounded by $\mathbb{E}\left[\frac{1}{1+\|S_1\|}\right]^{-1}$ and

$$\mathbb{E}_{\mathbb{Q}}[|S_1^i|] = \mathbb{E}\left[\frac{|S_1^i|}{1+\|S_1\|}\right] \Big/ \mathbb{E}\left[\frac{1}{1+\|S_1\|}\right] < \infty, \quad i = 1, \dots, d. \tag{11}$$

This shows that $\mathcal{Q} \neq \emptyset$. To show that $\mathcal{Q}$ is also convex, take $\mathbb{Q}^1, \mathbb{Q}^2 \in \mathcal{Q}$ and let $\alpha \in (0,1)$. Then $\mathbb{Q}^\alpha := \alpha\mathbb{Q}^1 + (1-\alpha)\mathbb{Q}^2$ is again a probability measure and equivalent to $\mathbb{P}$ on $\mathscr{F}_1$. Moreover, $\frac{\mathrm{d}\mathbb{Q}^\alpha}{\mathrm{d}\mathbb{P}}$ is bounded since $\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}}$ and $\frac{\mathrm{d}\mathbb{Q}^2}{\mathrm{d}\mathbb{P}}$ are so, and

$$\frac{\mathrm{d}\mathbb{Q}^\alpha}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}(\alpha\mathbb{Q}^1 + (1-\alpha)\mathbb{Q}^2)}{\mathrm{d}\mathbb{P}} = \alpha\frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}} + (1-\alpha)\frac{\mathrm{d}\mathbb{Q}^2}{\mathrm{d}\mathbb{P}} \leq \frac{\mathrm{d}\mathbb{Q}^1}{\mathrm{d}\mathbb{P}} + \frac{\mathrm{d}\mathbb{Q}^2}{\mathrm{d}\mathbb{P}}. \tag{12}$$

Finally,

$$\mathbb{E}_{\mathbb{Q}^\alpha}[|S_1^i|] = \mathbb{E}_{\alpha\mathbb{Q}^1+(1-\alpha)\mathbb{Q}^2}[|S_1^i|] = \alpha\mathbb{E}_{\mathbb{Q}^1}[|S_1^i|] + (1-\alpha)\mathbb{E}_{\mathbb{Q}^2}[|S_1^i|] < \infty, \quad i = 1, \dots, d, \tag{13}$$

which shows that $\mathbb{Q}^\alpha \in \mathcal{Q}$.

(b) Fix $\mathbb{Q} \in \mathcal{Q}$, and for $\epsilon \in (0,1)$, define the function

$$\varphi_\epsilon = \epsilon \mathbb{1}_{\{\vartheta^{tr}\Delta S_1 \geq 0\}} + (1-\epsilon)\mathbb{1}_{\{\vartheta^{tr}\Delta S_1 < 0\}} \tag{14}$$

and the measure $\mathbb{Q}_\epsilon \approx \mathbb{Q}$ on $\mathscr{F}_1$ by

$$\frac{\mathrm{d}\mathbb{Q}_\epsilon}{\mathrm{d}\mathbb{Q}} = \frac{\varphi_\epsilon}{\mathbb{E}_{\mathbb{Q}}[\varphi_\epsilon]}. \tag{15}$$

Then $0 < \varphi_\epsilon \leq 1$ for all $\epsilon \in (0,1)$ and hence $\mathbb{Q}_\epsilon \in \mathcal{Q}$ since $\mathbb{Q} \in \mathcal{Q}$ and $\frac{\mathrm{d}\mathbb{Q}_\epsilon}{\mathrm{d}\mathbb{Q}}$ is bounded. (Note that since $\mathscr{F}_0$ is trivial, S_0^i is a constant and hence integrability (under some measure) of S^i and ΔS^i is equivalent, $i = 1, \dots, d$.) Now suppose that there exists $\vartheta \in \mathbb{R}^d$ such that $\vartheta^{tr}x \geq 0$ for all $x \in \mathcal{C}$. Then in particular,

$$\mathbb{E}_{\mathbb{Q}_\epsilon}[\vartheta^{tr}\Delta S_1] = \frac{\mathbb{E}_{\mathbb{Q}}[\varphi_\epsilon \vartheta^{tr}\Delta S_1]}{\mathbb{E}_{\mathbb{Q}}[\varphi_\epsilon]} \geq 0. \tag{16}$$

Thus, by dominated convergence

$$\mathbb{E}_{\mathbb{Q}}[\vartheta^{tr}\Delta S_1 \mathbb{1}_{\{\vartheta^{tr}\Delta S_1 < 0\}}] = \lim_{\epsilon \to 0} \mathbb{E}_{\mathbb{Q}}[\varphi_\epsilon \vartheta^{tr}\Delta S_1] \geq 0. \tag{17}$$

But this implies that $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{Q}$-a.s. and, since $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$, also $\mathbb{P}$-a.s.

(c) Since $\mathscr{F}_0$ is trivial, we can identify any d-dimensional predictable process $\vartheta = (0, \vartheta_1)$ with the vector $\vartheta_1 \in \mathbb{R}^d$. For convenience and in a slight abuse of notation, write ϑ instead of ϑ_1. With this notation, it is straightforward to check that S satisfies NA if and only if whenever $\vartheta \in \mathbb{R}^d$ satisfies $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{P}$-a.s. then $\vartheta^{tr}\Delta S_1 = 0$ $\mathbb{P}$-a.s.

First, suppose that S satisfies NA. It suffices to show that the set $\mathcal{C}$ contains the origin. Seeking a contradiction, suppose that this is not the case. Then by the hint, there exist $\vartheta \in \mathbb{R}^d$ and $\mathbb{Q}_0 \in \mathcal{Q}$ such that $\vartheta^{tr}x \geq 0$ for all $x \in \mathcal{C}$ and $\mathbb{E}_{\mathbb{Q}_0}[\vartheta^{tr}\Delta S_1] > 0$. But then by part (b), it follows that $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{P}$-a.s. and, since $\mathbb{Q}_0 \approx \mathbb{P}$ on $\mathscr{F}_1$, also $\mathbb{Q}_0$-a.s. But this implies that $\mathbb{Q}_0[\vartheta^{tr}\Delta S_1 > 0] > 0$, and, since $\mathbb{Q}_0 \approx \mathbb{P}$ on $\mathscr{F}_1$, also $\mathbb{P}[\vartheta^{tr}\Delta S_1 > 0] > 0$, and we arrive at a contradiction.

Conversely, suppose that there exists $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ bounded and such that S is a $\mathbb{Q}$-martingale. This implies in particular that $\mathbb{E}_{\mathbb{Q}}[\vartheta^{tr}\Delta S_1] = 0$ for all $\vartheta \in \mathbb{R}^d$ and hence, whenever $\vartheta \in \mathbb{R}^d$ satisfies $\vartheta^{tr}\Delta S_1 \geq 0$ $\mathbb{P}$-a.s. then $\vartheta^{tr}\Delta S_1 = 0$ $\mathbb{P}$-a.s. since $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_1$.

Solution 2-4

(a) Since $A^1 - A^2$ is adapted, continuous, of finite variation and null at 0, it suffices to show that $A^1 - A^2$ is a local martingale. To this end, define for each $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,T] : \max(A_t^1, A_t^2) \geq n\} \wedge T. \tag{18}$$

Then for each $n \in \mathbb{N}$, the stopped processes $(A^1)^{\tau_n}$ and $(A^2)^{\tau_n}$ are both uniformly bounded by n, and $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} \mathbb{P}[\tau_n = T] = 1$. We proceed to show that for each $n \in \mathbb{N}$, the stopped process $(A^1 - A^2)^{\tau_n}$ is a uniformly integrable martingale. So fix $n \in \mathbb{N}$. It suffices to show that for each stopping time σ, $\mathbb{E}[(A^1)^{\tau_n}_\sigma] = \mathbb{E}[(A^2)^{\tau_n}_\sigma]$. So let σ be an arbitrary stopping time. Then

$$\begin{aligned}\mathbb{E}[(A^1)^{\tau_n}_\sigma] &= \mathbb{E}\left[\int_0^T \mathbb{1}_{]\!]0,\sigma\wedge\tau_n]\!]}\,\mathrm{d}A^1_s\right] = (\mathbb{P}\otimes A^1)\Big[]\!]0,\sigma\wedge\tau_n]\!]\Big]\\ &= (\mathbb{P}\otimes A^2)\Big[]\!]0,\sigma\wedge\tau_n]\!]\Big] = \mathbb{E}\left[\int_0^t \mathbb{1}_{]\!]0,\sigma\wedge\tau_n]\!]}\,\mathrm{d}A^2_s\right]\\ &= \mathbb{E}[(A^2)^{\tau_n}_\sigma].\end{aligned} \tag{19}$$

(b) Consider the σ-finite measures $\mathbb{P}\otimes B$ and $\mathbb{P}\otimes C$ on $(\bar{\Omega}, \mathcal{P})$. Then by the Lebesgue decomposition theorem, there exist unique σ-finite measures $\nu_a \ll \mathbb{P}\otimes C$ and $\nu_s \perp \mathbb{P}\otimes C$ on $(\bar{\Omega}, \mathcal{P})$ such that $\nu_a + \nu_s = \mathbb{P}\otimes B$. In particular, there exists a predictable set $\bar{N} \in \mathcal{P}$ such that $\nu_s = \mathbb{1}_{\bar{N}}(\mathbb{P}\otimes B)$, $\nu_a = \mathbb{1}_{\bar{\Omega}\setminus\bar{N}}(\mathbb{P}\otimes B)$ and $(\mathbb{P}\otimes C)[\bar{N}] = 0$. Define the processes $B^1 = (B^1_t)_{t\in[0,T]}$, $B^2 = (B^2_t)_{t\in[0,T]}$ and $C^1 = (C^1_t)_{t\in[0,T]}$ by

$$B^1_t = \int_0^t \mathbb{1}_{\bar{N}}\,\mathrm{d}B_s, \quad B^2_t = \int_0^t \mathbb{1}_{\bar{\Omega}\setminus\bar{N}}\,\mathrm{d}B_s \quad \text{and} \quad C^1_t := \int_0^t \mathbb{1}_{\bar{N}}\,\mathrm{d}C_s. \tag{20}$$

Then B^1, B^2 and C^1 are increasing, adapted, continuous and null at 0, $B^1 + B^2 = B$, $\nu_s = \mathbb{P}\otimes B^1$ and $\nu_a = \mathbb{P}\otimes B^2$, and for all $\bar{A} \in \mathcal{P}$,

$$(\mathbb{P}\otimes C^1)[\bar{A}] = \mathbb{E}\left[\int_{\bar{A}} \mathbb{1}_{\bar{N}}\,\mathrm{d}C_s\right] = (\mathbb{P}\otimes C)[\bar{A}\cap\bar{N}] = 0 = (\mathbb{P}\otimes 0)[\bar{A}], \tag{21}$$

where 0 denotes the zero process. Thus by part (a), we may deduce that $C^1 \equiv 0$. Next, by the Radon–Nikodým theorem, there exists a predictable process $H \geq 0$ such that

$$\nu_a = \mathbb{P}\otimes B^2 = H(\mathbb{P}\otimes C). \tag{22}$$

We proceed to show that $H \in L(C)$ and hence $H(\mathbb{P}\otimes C) = \mathbb{P}\otimes(H \bullet C)$. To this end, define for each $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,T] : \max(B^2_t, C_t) \geq n\} \wedge T. \tag{23}$$

Then for each $n \in \mathbb{N}$, the stopped processes $(B^2)^{\tau_n}$ and C^{τ_n} are both uniformly bounded by n, and $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} \mathbb{P}[\tau_n = T] = 1$. Hence, it suffices to show that $H \in L(C^{\tau_n})$ for all $n \in \mathbb{N}$. So fix $n \in \mathbb{N}$. Then by the definition of H,

$$\begin{aligned}\mathbb{E}\left[\int_0^T H_s\,\mathrm{d}C^{\tau_n}_s\right] &= \mathbb{E}\left[\int_0^T H_s \mathbb{1}_{]\!]0,\tau_n]\!]}\,\mathrm{d}C_s\right] = \int_{\bar{\Omega}} \mathbb{1}_{]\!]0,\tau_n]\!]} H\,\mathrm{d}(\mathbb{P}\otimes C)\\ &= \int_{\bar{\Omega}} \mathbb{1}_{]\!]0,\tau_n]\!]}\,\mathrm{d}(\mathbb{P}\otimes B^2) = \mathbb{E}\left[\int_0^T \mathbb{1}_{]\!]0,\tau_n]\!]}\,\mathrm{d}B^2_s\right] = \mathbb{E}[B^2_{\tau_n}] \leq n.\end{aligned} \tag{24}$$

Finally, since $\mathbb{P}\otimes(H \bullet C) = \mathbb{P}\otimes B^2$, it follows from part (a), that $B^2 = H \bullet C$, which together with the above establishes the claim.

(c) Suppose that S satisfies NA.

First, write $\mathrm{Var}(A) = (\mathrm{Var}(A)_t)_{t\in[0,T]}$,$A^+ = (A_t^+)_{t\in[0,T]}$ and $A^- = (A_t^-)_{t\in[0,T]}$ for the total, the positive and the negative variation of A, respectively. Then $\mathrm{Var}(A)$, A^+ and A^- are all increasing, adapted, continuous and null at 0, and $A = A^+ - A^-$ and $\mathrm{Var}(A) = A^+ + A^-$. On the level of measures, this means that

$$\mathbb{P}\otimes A^+, \mathbb{P}\otimes A^- \ll \mathbb{P}\otimes \mathrm{Var}(A) \quad \text{and} \quad (\mathbb{P}\otimes A^+) + (\mathbb{P}\otimes A^-) = \mathbb{P}\otimes \mathrm{Var}(A). \tag{25}$$

Hence there exist $\bar{D}^+ \in \mathcal{P}$ and $\bar{D}^- = \bar{\Omega}\setminus \bar{D}^+$ such that

$$\begin{aligned} \mathbb{P}\otimes A^+ &= \mathbb{1}_{\bar{D}^+}(\mathbb{P}\otimes \mathrm{Var}(A)) = \mathbb{P}\otimes(\mathbb{1}_{\bar{D}^+}\bullet \mathrm{Var}(A)),\\ \mathbb{P}\otimes A^- &= \mathbb{1}_{\bar{D}^-}(\mathbb{P}\otimes \mathrm{Var}(A)) = \mathbb{P}\otimes(\mathbb{1}_{\bar{D}^-}\bullet \mathrm{Var}(A)). \end{aligned} \tag{26}$$

By part (a), it follows that $A^+ = \mathbb{1}_{\bar{D}^+}\bullet \mathrm{Var}(A)$ and $A^- = \mathbb{1}_{\bar{D}^-}\bullet \mathrm{Var}(A)$.

Next, if there exist predictable processes $H^+, H^- \in L(\langle M\rangle)$ such that

$$A_t^+ = \int_0^t H_s^+ \,\mathrm{d}\langle M\rangle_s \quad \text{and} \quad A_t^- = \int_0^t H_s^- \,\mathrm{d}\langle M\rangle_s, \quad t\in[0,T], \tag{27}$$

we are done by setting $H := H^+ - H^-$. So, seeking a contradiction, assume without loss of generality that there does not exist $H^+ \in L(\langle M\rangle)$ such that $A^+ = \int H^+ \,\mathrm{d}\langle M\rangle$. Then by part (b), there exists $\widetilde{H}^+ \in L(\langle M\rangle)$ and $\bar{N}^+ \in \mathcal{P}$ such that

$$A_t^+ = \int_0^t \widetilde{H}_s^+ \,\mathrm{d}\langle M\rangle_s + \int_0^t \mathbb{1}_{\bar{N}^+}\,\mathrm{d}A_s^+ \quad \text{and} \quad \int_0^t \mathbb{1}_{\bar{N}^+}\,\mathrm{d}\langle M\rangle_s = 0, \quad t\in[0,T], \tag{28}$$

with $\mathbb{P}\left[\int_0^T \mathbb{1}_{\bar{N}^+}\,\mathrm{d}A_s^+ > 0\right] > 0$. (Otherwise, we could set $H^+ := \widetilde{H}^+$.) Define the strategy $\vartheta = (\vartheta_t)_{t\in[0,T]}$ by $\vartheta := \mathbb{1}_{\bar{N}^+}\mathbb{1}_{\bar{D}^+}$. Then $\vartheta \in L(S)$ as it is predictable and bounded, and it satisfies $\vartheta\bullet M \equiv 0$ as

$$\vartheta\bullet\langle M\rangle = (\mathbb{1}_{\bar{N}^+}\mathbb{1}_{\bar{D}^+})\bullet\langle M\rangle = \mathbb{1}_{\bar{D}^+}\bullet(\mathbb{1}_{\bar{N}^+}\bullet\langle M\rangle) = \mathbb{1}_{\bar{D}^+}\bullet 0 \equiv 0. \tag{29}$$

Moreover,

$$\begin{aligned} \vartheta\bullet A &= \vartheta\bullet A^+ - \vartheta\bullet A^-\\ &= (\mathbb{1}_{\bar{N}^+}\mathbb{1}_{\bar{D}^+})\bullet(\mathbb{1}_{\bar{D}^+}\bullet \mathrm{Var}(A)) - (\mathbb{1}_{\bar{N}^+}\mathbb{1}_{\bar{D}^+})\bullet(\mathbb{1}_{\bar{D}^-}\bullet \mathrm{Var}(A))\\ &= \mathbb{1}_{\bar{N}^+}\bullet\Big((\mathbb{1}_{\bar{D}^+}\mathbb{1}_{\bar{D}^+})\bullet \mathrm{Var}(A)\Big) - \mathbb{1}_{\bar{N}^+}\bullet\Big((\mathbb{1}_{\bar{D}^+}\mathbb{1}_{\bar{D}^-})\bullet \mathrm{Var}(A)\Big)\\ &= \mathbb{1}_{\bar{N}^+}\bullet(\mathbb{1}_{\bar{D}^+}\bullet \mathrm{Var}(A)) - \mathbb{1}_{\bar{N}^+}\bullet(0\bullet \mathrm{Var}(A))\\ &= \mathbb{1}_{\bar{N}^+}\bullet A^+ - \mathbb{1}_{\bar{N}^+}\bullet 0 = \mathbb{1}_{\bar{N}^+}\bullet A^+. \end{aligned} \tag{30}$$

Thus,

$$\vartheta\bullet S = \mathbb{1}_{\bar{N}^+}\bullet A^+ \geq 0 \quad \text{and} \quad \mathbb{P}[\vartheta\bullet S_T > 0] = \mathbb{P}\left[\int_0^T \mathbb{1}_{\bar{N}^+}\,\mathrm{d}A_s^+ > 0\right] > 0. \tag{31}$$

Thus ϑ is 0-admissible and S fails NA for 0-admissible strategies, in contradiction to the hypothesis.

Solution 2-5

(a) First, we show existence of a strong solution. To this end, note that $\mu \in L^2_{\text{loc}}((W_s)_{s\in[0,t]})$ for all $t < 1$ and define the process $\bar{Z} = (\bar{Z}_t)_{t\in[0,1)}$ by

$$\bar{Z}_t = \mathcal{E}\Big(-\int_0^{\cdot} \mu_s \,\mathrm{d}W_s\Big)_t = \exp\Big(-\int_0^t \mu_s \,\mathrm{d}W_s - \frac{1}{2}\int_0^t \mu_s^2 \,\mathrm{d}s\Big), \quad t < 1. \tag{32}$$

Then for fixed $t \in (0,1)$ the process $\bar{Z}$ restricted to $[0,t]$ is a strictly positive continuous local martingale and the unique strong solution on $[0,t]$ to the SDE

$$\mathrm{d}Z_t = -Z_t\mu_t \,\mathrm{d}W_t, \quad Z_0 = 1. \tag{33}$$

By Fatou's lemma, it follows that $\bar{Z}$ is a strictly positive supermartingale on $[0,t]$ for each fixed $t < 1$ and therefore also on $[0,1)$. By the supermartingale convergence theorem, $\lim_{t\uparrow 1} \bar{Z}_t$ exists $\mathbb{P}$-a.s. Define the process $Z = (Z_t)_{t\in[0,1]}$ by $Z_t := \bar{Z}_t$ for $t < 1$ and $Z_1 := \lim_{t\to 1} \bar{Z}_t$. Clearly Z is continuous, adapted and nonnegative. To show that it is a local martingale, define for $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \in [0,1) : Z_t > n\} \wedge 1. \tag{34}$$

Then $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty} \mathbb{P}[\tau_n = 1] = 1$. We proceed to show that for each fixed $n \in \mathbb{N}$, Z^{τ_n} is a uniformly integrable martingale on $[0,1]$. Since Z is a supermartingale, this is equivalent to showing that $\mathbb{E}[Z_{\tau_n}] = \mathbb{E}[Z_0] = 1$. To this end, note that Z^{τ_n} is uniformly bounded by n. Moreover, for each fixed $m \in \mathbb{N}$, $Z^{\tau_n \wedge \frac{m-1}{m}}$ is a bounded local and hence uniformly integrable martingale on $[0, \frac{m-1}{m}]$. This implies in particular that $\mathbb{E}[Z_{\tau_n \wedge \frac{m-1}{m}}] = 1$. Now dominated convergence shows that $\mathbb{E}[Z_{\tau_n}] = 1$.

To show that Z satisfies (33) on $[0,1]$, let $M \in \mathcal{H}_0^{2,c}$ be arbitrary. Then the fact that Z satisfies (33) on $[0,t]$ for each fixed $t \in (0,1)$ gives

$$\langle Z, M\rangle_t = \int_0^t -Z_s\mu_s \,\mathrm{d}\langle W, M\rangle_s, \quad t < 1,$$
$$\langle Z, Z\rangle_t = \int_0^t Z_s^2\mu_s^2 \,\mathrm{d}s, \quad t < 1,$$

Then monotone convergence gives

$$\int_0^1 Z_s^2\mu_s^2 \,\mathrm{d}s = \langle Z, Z\rangle_1 < \infty \ \ \mathbb{P}\text{-a.s.}, \tag{35}$$

and this together with the Kunita-Watanabe inequality and dominated convergence gives

$$\langle Z, M\rangle_1 = \int_0^1 -Z_s\mu_s \,\mathrm{d}\langle W, M\rangle_s \tag{36}$$

Since $M \in \mathcal{H}^{2,c}$ was arbitrary, Z solves (33) on $[0,1]$ by the definition of the stochastic integral.

Next, we show uniqueness. So suppose that Z^1 and Z^2 are solutions of (33) on $[0,1]$. Then they are a fortiori solutions of (33) on $[0,t]$ for each fixed $t \in (0,1)$. But for each fixed $t \in (0,1)$, the solution of (33) is unique, and so Z^1 and Z^2 coincide on $[0,t]$ for each fixed $t \in (0,1)$, and by continuity also on $[0,1]$.

(b) Let $\widetilde{Z} = (\widetilde{Z}_t)_{t\in[0,1]}$ be a local $\mathbb{P}$-martingale for the filtration $(\mathscr{F}_t^W)_{t\in[0,1]}$ with $\widetilde{Z}_0 = 1$ such that $\widetilde{Z}S$ is also local $\mathbb{P}$-martingale for the filtration $(\mathscr{F}_t^W)_{t\in[0,1]}$. By Itô's representation theorem, we may assume that $\widetilde{Z}$ has continuous paths and that there exists a predictable process $\widetilde{H} \in L^2_{\text{loc}}(W)$ such that

$$\widetilde{Z}_s = 1 + \int_0^t \widetilde{H}_s \, \mathrm{d}W_s, \quad t \in [0,1]. \tag{37}$$

Now the product rule gives

$$\widetilde{Z}_t S_t - \widetilde{Z}_0 S_0 = \int_0^t S_s \, \mathrm{d}\widetilde{Z}_s + \int_0^t \widetilde{Z}_s S_s \, \mathrm{d}W_s + \int_0^t \widetilde{Z}_s S_s \mu_s \, \mathrm{d}s + \int_0^t \widetilde{H}_s S_s \, \mathrm{d}s, \quad t \in [0,T]. \tag{38}$$

Since $\widetilde{Z}S - \widetilde{Z}_0 S_0$, $\int S \, \mathrm{d}\widetilde{Z}$ and $\int \widetilde{Z} S \, \mathrm{d}W$ are continuous local martingales null at 0, if follows that $\int (\widetilde{Z} S \mu + \widetilde{H} S) \, \mathrm{d}s$ is a continuous local martingale null at 0. Since it is of finite variation it must be constant 0. But this implies that for a.a. ω, $\widetilde{Z} S \mu + \widetilde{H} S$ is 0 a.e. on $[0,1]$. Since S is strictly positive the same is true for $\widetilde{Z}\mu + \widetilde{H}$. But his implies that $\int (\widetilde{Z}\mu + \widetilde{H})^2 \, \mathrm{d}s$ is constant 0 and hence $\int \widetilde{H} \, \mathrm{d}W = \int (-\widetilde{Z}\mu) \, \mathrm{d}W$, which shows that $\widetilde{Z}$ solves (33) on $[0,1]$. By uniqueness of the solution, we may deduce that $\widetilde{Z} = Z$.

(c) Define $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_1$ by $\mathrm{d}\mathbb{Q} := Z_1 \, \mathrm{d}P$. Note that since Z is strictly positive on $[0,1)$, $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_t$ for all $t \in (0,1)$. Moreover, S is a local $\mathbb{Q}$-martingale by part (b). It suffices to show that all $\vartheta \in \Theta_{\text{adm}}$ with $\vartheta \bullet S_1 \geq 0$ $\mathbb{P}$-a.s. satisfy $\vartheta \bullet S_1 = 0$ $\mathbb{P}$-a.s. So let $\vartheta \in \Theta_{\text{adm}}$ with $\vartheta \bullet S_1 \geq 0$ $\mathbb{P}$-a.s. Then by absolute continuity, $\vartheta \bullet S_1 \geq 0$ $\mathbb{Q}$-a.s. and hence $\vartheta \bullet S \equiv 0$ by the fact that $\vartheta \bullet S$ is a $\mathbb{Q}$-supermartingale (by Ansel-Stricker and Fatou) with $\vartheta \bullet S_0 = 0$. But since $\mathbb{Q} \approx \mathbb{P}$ on $\mathscr{F}_t$ for all $t \in [0,1)$, this implies that $\vartheta \bullet S_t = 0$ $\mathbb{P}$-a.s. for all $t \in [0,1)$, and continuity of $\vartheta \bullet S$ gives $\vartheta \bullet S_1 = 0$ $\mathbb{P}$-a.s.

(d) By the fundamental theorem of asset pricing, S satisfies NFLVR if and only if there exists a strictly positive $\mathbb{P}$-martingale $\widetilde{Z} = (\widetilde{Z}_t)_{t \in [0,1]}$ with $\widetilde{Z}_0 = 1$ such that $\widetilde{Z}S$ is a local $\mathbb{P}$-martingale. But if $\widetilde{Z}$ exists, then part b) shows that $\widetilde{Z} = Z$. This establishes the claim.

(e) First, note that the process $\widetilde{Z}$ is well defined by part (a) since

$$\int_0^1 \frac{1}{\sqrt{1-s}} \, \mathrm{d}s = 2 < \infty \quad \text{and} \quad \int_0^t \frac{1}{1-s} \, \mathrm{d}s = \log\left(\frac{1}{1-t}\right) < \infty, \quad t \in (0,1). \tag{39}$$

Next, note that $Z = \widetilde{Z}^\tau$ and $\sup_{t \in [0,1]} Z_t \leq 2$ $\mathbb{P}$-a.s., which shows that Z is a bounded local and hence true $\mathbb{P}$-martingale. Moreover, $Z_\tau = 0$ on $\{\tau = 1\}$ (since $\widetilde{Z}_1 = 0$ $\mathbb{P}$-a.s.) and $Z_\tau = 2$ on $\{\tau < 1\}$, which implies that $\mathbb{P}[Z_1 = 0] = \mathbb{P}[Z_1 = 2] = 1/2$ by the fact that

$$1 = \mathbb{E}[Z_1] = 0 \times \mathbb{P}[Z_1 = 0] + 2 \times \mathbb{P}[Z_1 = 2]. \tag{40}$$

Now the claim follows immediately from part (c) and part (d).