Topics in Discrete Mathematics

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Practice Problems

Solution of every problem should be no longer than one page!

Problem 1: Show that if $G$ is a graph with even degrees, the edges of $G$ can be oriented in such a way that every vertex in the resulting orientation has the same in-degree as out-degree.

Problem 2: Let $a_1, a_2, \ldots, a_n$ be $n$ not necessarily distinct integers. Show there is a set of consecutive numbers $a_k, a_{k+1}, \ldots, a_\ell$ whose sum $\sum_{i=k}^\ell a_i$ is divisible by $n$.

Problem 3: Prove that for every $k \geq 2$ there exists an $n_0 = n_0(k)$ such that every colouring of $1, 2, \ldots, n_0$ in $k$ colours contains three distinct numbers $1 \leq a, b, c \leq n_0$ satisfying $a \cdot b = c$ that have the same colour.

Problem 4: Prove that for every positive integer $r$ there exists $N(r)$ such that for all $n \geq N(r)$, any colouring of all subsets of $[n]$ into $r$ colours contains two non-empty disjoint sets $X$ and $Y$ such that $X$, $Y$ and $X \cup Y$ have the same colour.

Problem 5: A transitive tournament is an orientation of a complete graph for which the vertices can be numbered in such a way that $(i,j)$ is a directed edge if and only if $i < j$.

(i) Show that every orientation of the complete graph $K_n$ contains a transitive tournament on $\lceil \log_2 n \rceil$ vertices.

(ii) Show that if $k \geq 2 \log_2 n + 2$ there is an orientation of $K_n$ with no transitive tournament on $k$ vertices.

Problem 6: Let $g_1(x), \ldots, g_k(x)$ be bounded real functions and let $f(x)$ be another real function. Suppose there are positive constants $\varepsilon$ and $\delta$ such that if $f(x) - f(y) > \varepsilon$, then $\max_i (g_i(x) - g_i(y)) > \delta$. Prove that $f$ is also bounded.

Problem 7: Prove that every set of $2^m + 1$ vectors in $\mathbb{R}^m$ with integer coordinates contains a pair of vectors whose average also has integer coordinates.
Problem 8: Let $G$ be a graph with $n$ vertices and $m$ edges. Show that $G$ has at least
\[
\frac{4m}{3n} \left( m - \frac{n^2}{4} \right)
\]
triangles. Moreover, show this estimate is tight (best possible) when $m = n^2/3$.

Problem 9: Let $G$ be a graph on $n$ vertices and let $\overline{G}$ be its complement. Let $t(G)$ denote the total number of triangles in $G$ and $\overline{G}$. Express $t(G)$ as a function of the degrees $d_1, \ldots, d_n$ of the vertices of $G$ and prove that
\[
t(G) \geq \frac{n(n-1)(n-5)}{24}.
\]

Problem 10: Suppose $r \geq 3$, $n \geq r + 1$, and let $\text{ex}(n, K_r)$ denote the maximum number of edges in a graph on $n$ vertices that does not contain $K_r$ as a subgraph. Show that any graph $G$ on $n$ vertices with at least 
\[
\text{ex}(n, K_r) + 1
\]
edges must contain $K_{r+1} - e$; i.e., a copy of a clique of size $r + 1$ with one missing edge.

Problem 11: Let $H$ be an $r$-uniform (edge edge has size $r$) hypergraph on $n$ vertices with $m = cn^{r-1}/e^{r-1}$ edges. Show that for a sufficiently large constant $c$, $H$ contains a collection of $r$ disjoint sets $U_1, \ldots, U_r$ such that $|U_i| = t$ for all $i$ and all $r$-tuples intersecting each $U_i$ in one vertex are edges of $H$. (That is, $H$ contains a complete $r$-partite subhypergraph with parts of size $t$.)

Problem 12: Let $X$ be a set of $n$ points in the plane. Prove that the number of pairs $x_i, x_j \in X$ such that the distance between $x_i$ and $x_j$ equals 1 is at most $cn^{3/2}$ for some absolute constant $c$.

Problem 13: Let $D$ be a directed graph on $n$ vertices such that the outdegree of every vertex is larger than $\log_2 n$. Prove that $D$ contains an even directed cycle.

[Hint: Show that the vertices of $D$ can be partitioned into two parts such that every vertex from one part has an out-neighbour in the other.]

Problem 14: Let $\mathcal{F}$ be a collection of subsets of $X$ such that every two members of $\mathcal{F}$ intersect in at least two points. Prove that the vertices of $X$ can be 2-coloured so that no set in $\mathcal{F}$ is monochromatic.
Problem 15: Let $k \geq 4$ and let $\mathcal{F}$ be a collection of $k$-element subsets of $X$. Prove that if $\mathcal{F}$ has fewer than $\frac{4^{k-1}}{3^k}$ sets then the vertices of $X$ can be 4-coloured so that every set in $\mathcal{F}$ is rainbow; i.e., contains vertices of all 4 colours.

Problem 16: Let $G$ be a graph with average degree at least $2d$. Prove that $G$ contains a non-empty subgraph $G'$ with minimum degree at least $d$. Using this, show that if $G$ has $n$ vertices and $kn$ edges, then it contains every tree on $k$ vertices as a subgraph (a tree is a connected graph with no cycles).

Problem 17: Given a tournament $T$, a Hamiltonian path is a directed path that visits every vertex of $T$ exactly once.

(i) Prove that every tournament contains a Hamiltonian path.

(ii) Prove there exists a tournament $T$ on $n$ vertices which contains at least $n!2^{-(n-1)}$ distinct Hamiltonian paths.

Problem 18: Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$ of unit length $|v_i| = 1$. Prove there are signs $\varepsilon_i = \pm 1$ such that

$$|\varepsilon_1 v_1 + \ldots + \varepsilon_n v_n| \leq \sqrt{n}.$$ 

Show that this is tight; i.e., the $\sqrt{n}$ estimate cannot be improved.

Problem 19: Given a hypergraph $H$, the transversal number $\tau(H)$ is the minimal cardinality of a set of vertices which intersects all edges of $H$. Prove that if $H$ has $n$ vertices and $m$ edges all of size $r$, then for any $p \in [0, 1]$,

$$\tau(H) \leq pn + (1-p)^r m.$$ 

Deduce from this that

$$\tau(H) \leq \frac{m + n \log r}{r},$$

where $\log$ is the natural log to the base $e$.

Problem 20: Let $\{(A_i, B_i) : 1 \leq i \leq h\}$ be a family of pairs of subsets of the set of integers such that $|A_i| = k$ and $|B_i| = \ell$ for all $i$, $A_i \cap B_i = \emptyset$ and $(A_i \cap B_j) \cup (A_j \cap B_i) \neq \emptyset$ for all $i \neq j$. Prove that $h \leq \frac{(k+\ell)^{k+\ell}}{k^k \ell^\ell}$. 

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Problem 21: A sunflower with \( k \) petals and core \( Y \) is a collection of sets \( S_1, \ldots, S_k \) such that \( S_i \cap S_j = Y \) for all \( i \neq j \) and the sets \( S_i - Y \) are all non-empty. Let \( \mathcal{F} \) be a family of sets each of cardinality \( s \). Prove that if \( |\mathcal{F}| > s!(k-1)^s \) then \( \mathcal{F} \) contains a sunflower with \( k \) petals.

Problem 22: Let \( G_1 \) and \( G_2 \) be two graphs on the same vertex set \( V \). Prove that the chromatic number of the union \( G_1 \cup G_2 \) (we take the union of the edge sets of both graphs) satisfies
\[
\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2).
\]
Use this to show that if \( H_1, \ldots, H_t \) are bipartite graphs whose union is a complete graph on \( n \) vertices, then \( t \geq \log_2 n \).

Problem 23: Let \( A_1, \ldots, A_m \) be subsets of an \( n \)-element set. Assume that their pairwise symmetric differences \( A_i \Delta A_j = (A_i - A_j) \cup (A_j - A_i) \) have only two sizes. Prove that \( m \leq \frac{n(n+1)}{2} + 1 \). Find \( m = \frac{n(n-1)}{2} + 1 \) subsets of an \( n \)-element set with only two sizes of symmetric differences.

Problem 24: Let \( \mathcal{A} \) and \( \mathcal{B} \) be families of subsets of an \( n \)-element set with the property that \( |A \cap B| \) is odd for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). Prove that \( |\mathcal{A}| |\mathcal{B}| \leq 2^{n-1} \).

Problem 25: Let \( \lambda_1 \) be the maximum eigenvalue of a graph \( G \). Prove that the chromatic number of \( G \) is at most \( \lambda_1 + 1 \).

Problem 26: Suppose that a connected graph \( G \) has only two distinct eigenvalues. Prove that \( G \) is the complete graph.

[ Hint (to be read backwards): ?hparg eht fo retemaid eht tuoba yas uoy nac tahW ]

Problem 27: Let \( G \) be a graph on \( n \) vertices with \( m \) edges. Let \( L(G) \) be the line graph of \( G \).

(i) Show that all eigenvalues of \( L(G) \) satisfy \( \lambda_i \geq -2 \).

(ii) Show that if \( m > n \), then the smallest eigenvalue satisfies \( \lambda_m = -2 \).