9) $\Delta u + \lambda u = 0$ in $G = [0,2\pi] \times [0,\pi]$

$u = 0$ auf ∂G

Ansatz: $u(x,y) = X(x)\,Y(y)$

$$\Delta u(x,y) + \lambda u = X''(x)Y(y) + X(x)Y''(y) + \lambda X(x)Y(y) \overset{!}{=} 0$$

$$(X''(x) + \lambda X(x))Y(y) = -X(x)Y''(y)$$

$$\frac{X''(x)+\lambda X(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \overset{!}{=} -\omega^2 \qquad \omega \in \mathbb{C}$$

$$Y(y) = A\cdot e^{\omega y} + B\cdot e^{-\omega y}$$

$$X(x) = C\cdot e^{\sqrt{-\omega^2-\lambda}\,x} + D e^{-\sqrt{-\omega^2-\lambda}\,x}$$

$u=0$ ist Lösung, betrachte $u \neq 0$; d.h. $X \neq 0$ und $Y \neq 0$:

$u(0,y) = 0 = (C+D)\,Y(y) \implies C = -D$

$u(x,0) = 0 = X(x)\,(A+B) \implies A = -B$

$\implies X(x) = E\cdot \sin(i\sqrt{-\omega^2-\lambda}\,x)$,

$Y(y) = F\cdot \sin(i\omega y)$

$u(2\pi,y) = 0 = E\cdot\sin(\sqrt{-\omega^2-\lambda}\,2\pi)\,Y(y) \implies i\sqrt{-\omega^2-\lambda}\,2\pi \in \pi\mathbb{Z}$

$\implies i\sqrt{-\omega^2-\lambda} = \frac{k}{2} \implies \lambda = \frac{k^2}{4} - \omega^2$

$u(x,\pi) = 0 = X(x)\,F\cdot\sin(i\omega\pi) \implies i\omega \in \mathbb{Z}$

$\implies \omega^2 = l^2(-1)$

Eigenwerte: $\underline{\underline{\lambda_{k,l} = l^2 + \frac{k^2}{4}}}$

Eigenvektoren: $\underline{\underline{u_{k,l} = \sin\left(\frac{k}{2}x\right)\sin(l_y)}}$