Serie 7

1. Take a (7×4) matrix $\mathbf{A} = (a_{ij})_{\substack{1\leq i\leq 7 \\ 1\leq j\leq 4}}$. With $\mathbf{A}$ we define a system of linear equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & a_{14}x_4 & = & 0\,, \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & a_{24}x_4 & = & 0\,, \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & a_{34}x_4 & = & 0\,, \\
a_{41}x_1 & + & a_{42}x_2 & + & a_{43}x_3 & + & a_{44}x_4 & = & 1\,, \\
a_{51}x_1 & + & a_{52}x_2 & + & a_{53}x_3 & + & a_{54}x_4 & = & 0\,, \\
a_{61}x_1 & + & a_{62}x_2 & + & a_{63}x_3 & + & a_{64}x_4 & = & 0\,, \\
a_{71}x_1 & + & a_{72}x_2 & + & a_{73}x_3 & + & a_{74}x_4 & = & 0\,.
\end{array}$$

For each of the following statements determine whether they are true or false.

(a) The solution space is a (possibly empty) subset of $\mathbb{R}^7$.

(b) The solution space is a (possibly empty) subset of $\mathbb{R}^4$.

(c) Such a linear system can have infinitely many solutions

(d) Such a linear system can have only one solution.

(e) This system always admits at least the trivial solution $x = 0$.

(f) The solution space is a (possibly zero-dimensional) subspace.

(g) There is a solution if the right hand side is in the column space of A.

(h) If the augmented matrix has rank 5, a solution exists.

(i) The set of differences between two solutions is a vector space.

(j) The arithmetic mean of two solutions is again a solution.

(k) If we delete the last three equations, the remaining system admits at least one solution.

(l) If we delete the last four equations, the remaining system admits non-trivial solutions.

2. Consider two vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{u},\ \mathbf{v} \neq \mathbf{0}$. We define an $n \times n$ matrix $\mathbf{A}$ via

$$\mathbf{A} := \mathbf{u}\mathbf{v}^\top .$$

Solve the following problems in the case of specific vectors $\mathbf{u}, \mathbf{v}$ (a) and in the general case (b).

(i) Which vectors span the row space of $\mathbf{A}$?
(ii) Which vectors span the column space of $\mathbf{A}$?
(iii) Determine the row echelon form of $\mathbf{A}$.
(iv) Determine the $\operatorname{rank}(\mathbf{A})$ and $\operatorname{im}(\mathbf{A})$.
(v) Describe $\ker(\mathbf{A})$ through an equation of a hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}, \mathbf{x} \rangle = 0\}$ with a suitable vector $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.
(vi) Determine the basis of $\operatorname{im}(\mathbf{A})$.

Find the solutions to problems (i)-(vi) for

(a) Vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ und $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ in $\mathbb{R}^4$.
(b) Arbitrary vectors $\mathbf{u},\ \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{u},\ \mathbf{v} \neq \mathbf{0}$.

3. Consider the matrix $\mathbf{A} \in \mathbb{R}^{n,m}$ for $n, m \in \mathbb{N}$. Let us add an additional column to $\mathbf{A}$, namely the vector $\mathbf{b} \in \mathbb{R}^n$. In other words, we consider the matrix $\mathbf{B} = [\mathbf{A}\,\mathbf{b}] \in \mathbb{R}^{n,m+1}$.

(a) Given an example of $\mathbf{A} \in \mathbb{R}^{4,3}$ and $\mathbf{b} \in \mathbb{R}^4$ for which the image of $\mathbf{A}$ is equal to the image of $\mathbf{B}$.
(b) Given an example of $\mathbf{A} \in \mathbb{R}^{4,3}$ and $\mathbf{b} \in \mathbb{R}^4$ for which the image of $\mathbf{B} = [\mathbf{A}\,\mathbf{b}] \in \mathbb{R}^{4,4}$ is strictly bigger than the image of $\mathbf{A}$.
(c) Specify general conditions under which the image of $\mathbf{B}$ is strictly larger than the image of $\mathbf{A}$. What does this mean for the columns of $\mathbf{A}$ and our vector $\mathbf{b}$?

4. We are given an invertible matrix $\mathbf{A} \in \mathbb{R}^{n,n}$. Furthermore, take an arbitrary $k \in \mathbb{N}$ and consider the set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\} \subset \mathbb{R}^n$.

(a) Show that the following statements are equivalent
  (i) The set of vectors $\{\mathbf{A}\mathbf{v}_1, \cdots, \mathbf{A}\mathbf{v}_k\}$ is linearly independent.
  (ii) The set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly independent.
(b) Show that for an arbitrary matrix $\mathbf{B} \in \mathbb{R}^{n,m}$ we have:

$$\ker(\mathbf{AB}) = \ker(\mathbf{B}) .$$

5. Consider two matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbb{R}^{n,n}$. Show that the following statements hold

(a) $\operatorname{im}(\mathbf{AB}) \subset \operatorname{im}(\mathbf{A})$.
(b) $\operatorname{rank}(\mathbf{AB}) \leq \operatorname{rank}(\mathbf{A})$.

(c) $\operatorname{rank}(\mathbf{B}^\top \mathbf{A}^\top) \leq \operatorname{rank}(\mathbf{A}^\top)$.

Furthermore, let us assume that

$$\mathbf{AB} = \mathbf{I}_n \ (\hat{=}\, n \times n \text{ identity matrix}). \tag{1}$$

Show that

(d) $\operatorname{rank}(\mathbf{A}) = n$.

Hint: You can use previously proven statements when proving later ones!
Comment: In this problem, we show that for two $n \times n$ matrices, for which (1) holds, we have $\operatorname{rank}(\mathbf{A}) = n$. This condition (d) implies that $\mathbf{A}$ is an invertible matrix. Thus we can infer from condition (1), not only that $\mathbf{B}$ is the right inverse of $\mathbf{A}$, but furthermore that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n \ .$$

That is, we conclude that $\mathbf{B}$ is a (two-sided) inverse of $\mathbf{A}$.

6. We consider the space $\mathcal{M} = \mathbb{R}^{n,n}$ of all $n \times n$ matrices with real valued entries. Analogously to $\mathbb{R}^n$, in this vector space $\mathcal{M}$ we define componentwise addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{R}^{n,n} \quad \text{mit} \quad c_{ij} := a_{ij} + b_{ij} \ , \quad i,j \in \{1, \ldots, n\} \ ,$$

and componentwise scalar multiplication

$$\mathbf{C} = \alpha \mathbf{A} \quad \text{mit} \quad c_{ij} := \alpha a_{ij} \ , \quad i,j \in \{1, \ldots, n\} \ .$$

Let us now conisder the following subsets of $\mathcal{M}$:

(a) The set of all diagonal matrices

$$\mathcal{D} := \left\{ \mathbf{A} = \{a_{ij}\}_{1 \leq i,j \leq n} \in \mathbb{R}^{n,n} \ : \ a_{ij} = 0 \text{ für alle } i \neq j \right\} ,$$

(b) The set of all upper-triangular matrices

$$\mathcal{R} := \left\{ \mathbf{A} = \{a_{ij}\}_{1 \leq i,j \leq n} \in \mathbb{R}^{n,n} \ : \ a_{ij} = 0 \text{ für alle } i > j \right\} ,$$

(c) The set of all symmetric matrices

$$\mathcal{S} := \left\{ \mathbf{A} = \{a_{ij}\}_{1 \leq i,j \leq n} \in \mathbb{R}^{n,n} \ : \ a_{ij} = a_{ji} \text{ für alle } 1 \leq i,j \leq n \right\} .$$

Show that the sets $\mathcal{D}, \mathcal{R}$ and $\mathcal{S}$ are subspaces of the governing vector space $\mathcal{M}$, and determine a basis for each one of those subspaces. Determine the dimensions of spaces $\mathcal{D}, \mathcal{R}$ and $\mathcal{S}$.

- **Abgabe der Serien:** Donnerstag, 14.11.2013 in der Übungsgruppe oder bis 16:00 Uhr in den Fächern im Vorraum zum HG G 53. Die Serien müssen sauber und ordentlich geschrieben und zusammengeheftet abgegeben werden, sofern eine Korrektur gewünscht wird.
- **Semesterpräsenz:** Montag, 15:15 - 17:00 Uhr, ETH Zentrum, LFW E 13. Falls keine grosse Nachfrage besteht, warten die Assistenten maximal eine halbe Stunde. Wir bitten Sie deshalb, bei Fragen so früh als möglich zu erscheinen.
- **Homepage:** Hier werden zusätzliche Informationen zur Vorlesung und die Serien und Musterlösungen als PDF verfügbar sein.
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