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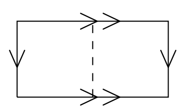
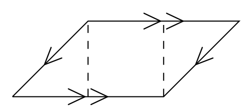
Algebraic Topology I
HS 2014

Problem set 4

Covering spaces- Homology

1. Use covering theory to prove the Nielsen Schreier Theorem: every subgroup of the free group $\mathbb{F}_2$ is free.

2. Let us consider the covering spaces of the torus $X = T^2$:

$\overline{X}_1$  $\overline{X}_2$ 

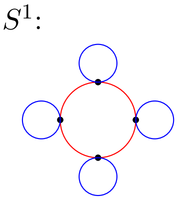
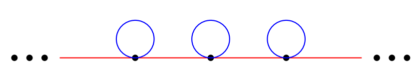
The squares in the pictures have sides of lenght 1 and the covering projections are given by the quotient by the equivalent relation $(x, y) \equiv (x + k, y)$ with k in $\mathbb{Z}$.

   (a) Show that $\overline{X}_i$ are homeomorphic topological spaces, and the described projections give covering spaces of the torus. Determine the number of sheets.

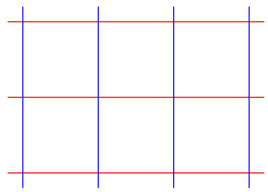
   (b) Find a loop in X whose lifts in $\overline{X}_1$ and $\overline{X}_2$ are not homeomorphic. Deduce that $\overline{X}_1$ and $\overline{X}_2$ are not isomorphic as covering spaces.

   (c*) To which subgroups of $\mathbb{Z}^2 = \pi_1(X, x)$ correspond the two covering spaces?

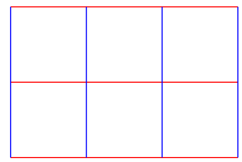
3. Find the subgroups of $\mathbb{F}_2$ corresponding to the following coverings of $S^1 \vee S^1$:

X_1  X_2 

X_3



X_4



The space X_3 is understood to be extending infinitely in all four directions (it is a grid in $\mathbb{R}^2$, in the graph X_4 the two vertical outermost segments and the two horizontal outermost segments are identified: it is a graph drawn on the torus with two horizontal segments and three vertical segments.

4. Denote by S^2 the two sphere, by T^2 the torus $S^1 \times S^1$, by K the Klein bottle.

   (a) Define a Delta complex structure on S^2, T^2, K consisting of two simplices.

   (b) How many simplices do you need to define a simplicial structure on S^2?

   (c) Compute the Delta complex homology for the three spaces.

5*. Construct a three dimensional Δ-complex X from 4 tetrahedral $T_1, \ldots, T_4$ by first arranging the tetrahedral in a cyclic pattern as in the figure so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show that the Δ-complex homology groups of X are $H_0^\Delta(X) = \mathbb{Z}, H_1^\Delta(X) = \mathbb{Z}/4\mathbb{Z}, H_2^\Delta(X) = 0, H_3^\Delta(X) = \mathbb{Z}$. The space X is an example of a *lens space*.