Prof. Rahul Pandharipande
ETH Zürich

Algebraic Topology I
HS 2014

Problem set 6

Homology, Hurewitz homomorphism

1. Prove that if R is a retract of X then each group $H_n(X)$ contains a subgroup isomorphic to $H_n(R)$ and in fact $H_n(X)$ can be expressed as the product $H_n(X) \cong H_n(R) \times K_n$. Give an example of a retract in which K_n is nonzero for some n.

2. A singular 1-simplex $\sigma : \Delta^1 \to X$ is called a *loop* if $\sigma(v_0) = \sigma(v_1)$.

   (a) Prove that a loop is a 1-cycle.

   (b) Two loops σ_0, σ_1 are *freely homotopic* if there exists a continuous map $F : [0,1]^2 \to X$ such that $F(0,t) = \sigma_0(t), F(1,t) = \sigma_1(t), F(s,0) = F(s,1)$. Prove that two freely homotopic loops are homologous, and that the constant path is null homologous.

   (c) Fix a basepoint $x \in X$. Define a map $\rho : \pi_1(X,x) \to H_1(X)$ and prove that it is a well defined homomorphism. Deduce that there exists a map $\overline{\rho} : \pi_1(X,x)^{\mathrm{ab}} \to H_1(X)$.

   (d) A 1-chain $\sigma_0 + \ldots + \sigma_{r-1}$ with $\sigma_i(v_0) = \sigma_{i-1}(v_1)$ for all $i \in \mathbb{Z}/_{r\mathbb{Z}}$ is called an *elementary 1-cycle*. Prove that an elementary 1-cycle is a 1-cycle, and it is homologous to a loop.

   (e) Prove that the classes of loops generate $H_1(X)$.

   (f) Assume that X is path-connected. Show that ρ is surjective

   Remark: the map $\overline{\rho}$ defined here is known as the first Hurewitz homomorphism, and it is possible to show that in this case, it is an isomorphism.

3. Let $f : (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_1 : H_1(X) \to H_1(Y)$.

Prove commutativity of the diagram

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[d, "\rho_X"] & \pi_1(Y,y_0) \arrow[d, "\rho_Y"] \\ H_1(X) \arrow[r, "f_1"] & H_1(Y) \end{tikzcd}$$

where ρ_X and ρ_Y are the homomorphisms defined in Exercise 2.

4. Let $p : X \to Y$ be a covering map, and let $x_0 \in X$ and $y_0 = p(x_0)$. We have proven in class that $p_* : \pi_1(X,x) \to \pi_1(Y,y)$ is injective. Is it true in general that $p_* : H_1(X) \to H_1(Y)$ is injective?

5. Let X be a Δ complex. The simplicial homology of X with real coefficient $\mathrm{H}_i^\Delta(X,\mathbb{R})$ is computed as the ordinary simplicial homology but taking combination with real coefficients instead of integer coefficients. In particular the n-chains with real coefficients are a real vectorspace, and the boundary maps are linear maps. The Euler characteristic of X is defined as

$$\chi(X) = \sum (-1)^i \dim \mathrm{H}_i^\Delta(X,\mathbb{R}).$$

Show that $\chi(X) = \sum(-1)^i \dim \mathrm{C}_i^\Delta(X,\mathbb{R})$. Compute the Euler characteristic of the surface of genus 2 (that is obtained from the regular octagon identifying parallel sides) and of the three dimensional torus $S^1 \times S^1 \times S^1$